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Prime values of meromorphic functions and irreducible polynomials

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Abstract: The existence of prime values of meromorphic functions and irreducible polynomials is considered. An argument for an infinite number of prime values of irreducible polynomials over $\mathbb{Z}[x]$ is provided.
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1 Meromorphic functions

The distributional characteristics of meromorphic functions might be used to model the entropic measure. For example, the function $\frac{x^y-1}{x-1}$ is irreducible for a set of prime integers y in $\mathbb{Q}[x]$. By analogy with the Hilbert irreducibility theorem, it can be conjectured that there exists an infinite number of values of x such that the polynomial

$$1 + x + ...x^{n-1} = \frac{x^n - 1}{x - 1}$$

is a prime for those prime values of n. The values of n, for fixed x, such that $\frac{x^n-1}{x-1}$ is prime, may be plotted vertically, while values of x for given n, such that the polynomial is prime, can be plotted horizontally. It may be hypothesized that the function $\frac{x^y-1}{x-1}$ has an infinite number of prime values only if x is integer and $y \geq 2$ is a fixed prime, or $x \geq 2$ is a fixed integer and y is integer. The density of prime values on each of these lines in the quadrant $\{(x, y)|x \geq 0, y \geq 0\}$ can be determined and compared to information-theoretic measures defined for sequences.

2 Irreducible polynomials

Let $f(x)$ be irreducible of degree n with coefficients that do not have a common prime factor. Suppose that the solutions to $f(x) \equiv 0 \ (mod \ p)$ are $w_{1,p}, ..., w_{v(p),p}$, where $0 \leq v(p) \leq n$. Let $F(1; p_1, ..., p_k)$ be the number of values x, $1 \leq \leq \xi$, such that $x \not\equiv w_{u,p_r} \ (mod \ p_r)$, $1 \leq r \leq k$, $1 \leq u \leq v(p_r)$. It has been demonstrated [1] that

$$\frac{\alpha_2}{log \ \eta} \leq \prod_{n \leq p \leq \eta} \left(1 - \frac{v(p)}{p}\right) \leq \frac{\alpha_3}{log \ \eta} \tag{2.1}$$

and

$$\begin{aligned} F(1; p_1, ..., p_k) \leq F(1) + \sum_{i=1}^{2t} (-1)^i \sum_{r_1, ..., r_i} v(p_{r_1})...v(p_{r_i}) F(p_{r_1}...p_{r_i}) \\ + \sum_{r_1, ..., r_i} v(p_{r_1})...v(p_{r_{2t}}) F(p_{r_1}....p_{r_{2t}}; p_1, ..., p_{Min(r_{2t-1}, k_t)}) \end{aligned} \tag{2.2}$$

where $k = k_0 > .... > k_t$, $r_j \leq k_{\left[\frac{j-1}{2}\right]}$ and $10n \leq \alpha_4 < p_1 < ... < p_k \leq \rho$ [1].

Defining

$$\tau = 1 + \sum_{i=1}^{2t} (-1)^i \sum_{r_1, ..., r_i} \frac{v(p_{r_1})}{p_{r_1}} ... \frac{v(p_{r_i})}{p_{r_i}} \tag{2.3}$$

it follows that

$$\begin{aligned} F(1; p_1, ..., p_k) &\geq \xi + \sum_{i=1}^{2t} (-1)^i \sum_{r_1, ..., r_i} v(p_{r_1})...v(p_{r_i}) \frac{\xi}{p_{r_1}...p_{r_i}} - 1 \\ &\quad - \sum_{i=1}^{2t} \sum_{r_1, ..., r_i} v(p_{r_1})...v(p_{r_i} \\ &\geq \xi\tau - 1 - \sum_{i=1}^{2t} \sum_{r_1, ..., r_i} n^i \\ &\geq \xi\tau - n \prod_{m=0}^{t-1} (nk_m)^2 \end{aligned} \tag{2.4}$$

for $n \geq 2$ since $\left| F(d) \ \frac{\xi}{d} \right| \leq 1$. It can be shown that $\tau < 2 \prod_{m=1}^{t} L_m$ [2], where

$$L_m = \prod_{k_m < s \leq k_{m-1}} \left(1 - \frac{v(p_s)}{p_s}\right), ..., 1 \leq m \leq t.$$

Supposing the inequality $\tau > \varepsilon_0 \prod_{m=1}^{t} L_m$ is valid also, based on the form of τ, for some constant ε_0. Similarly, $\prod_{m=0}^{t-1} (nk_m)^2 \leq \xi^{\frac{2}{3}}$ [2]. If a sieve method is chosen such that k is considerably less than the maximal number given by $\frac{\sqrt{p}}{log \ \sqrt{p}}$, then the number of prime values of $f(x)$ for $x \leq \xi$ is

$$
\begin{aligned}
P(\xi) &\geq F(1; p_1, ..., p_k) \\
&\geq \xi\varepsilon_0 \prod_{\alpha_4 \leq p \leq \rho} \left(1 - \frac{v(p)}{p}\right) - 1 - n\xi^{\frac{2}{3}} \\
&\geq \xi \cdot \varepsilon_0 \alpha_9 \prod_{n < p \leq \rho} \left(1 - \frac{v(p)}{p}\right) - 1 - n\xi^{\frac{2}{3}} \\
&\geq \xi \cdot \varepsilon \alpha_9 \cdot \frac{\alpha_2}{log\ \rho} - 1 - n\xi^{\frac{2}{3}}.
\end{aligned}
\tag{2.5}
$$

If $x \leq \xi$ and $f(x)$ is positive and monotonically increasing for sufficiently large x, $f(k) \leq f(\xi)$. Therefore, ρ can be substituted by $\sqrt{f(\xi)}$ and $log\sqrt{f(\xi)} \sim \frac{1}{2}n\ log\ \xi$.

As $\dfrac{\alpha_2}{log_\rho} > \dfrac{\alpha_2}{\frac{1}{2}n\alpha_{11} log\ \xi}$,

$$
\begin{aligned}
P(\xi) &\geq \frac{2\varepsilon_0 \cdot \alpha_9 \cdot \alpha_2}{n\alpha_{11}} \frac{\xi}{log\ \xi} - 1 - n\xi^{\frac{2}{3}} \\
&\geq \left(2\frac{\varepsilon_0 \cdot \alpha_9 \cdot \alpha_2}{n\alpha_{11}} - \alpha_{12}\right) \frac{\xi}{log\ \xi} = \alpha_{13} \frac{\xi}{log\ \xi}.
\end{aligned}
\tag{2.6}
$$

A proof of the second inequality for would be sufficient to establish the existence of an infinite number of prime values of the irreducible polynomial.

3 A lower bound for the number of prime values of a primitive irreducible polynomial

Given a number ξ, and a primitive irreducible polynomial $f(x)$, with $v(p)$ solutions to the congruence $f(x) \equiv 0\ (mod\ p)$, the fraction $1 - \frac{v(p)}{p}$ represents the relative number of values of x which have the property $p \not| f(x)$. Let p range from $2to\xi$. Then the minimum number for values of x such that $f(x)$ is not divisible by any prime between 2 and ξ is

$$
\xi \prod_{2 \leq p \leq \xi} \left(1 - \frac{v(p)}{p}\right) \geq \xi \cdot \frac{\alpha_2}{log\ \xi}. \tag{3.1}
$$

Therefore, both an upper and a lower bound for the number of values of x such that $f(x)$ is prime for $x \leq \xi$ satisfies

$$
\alpha_2 \frac{\xi}{log\ \xi} < P(\xi) < \alpha_1 \frac{\xi}{log\ \xi} \tag{3.2}
$$

where the upper bound is derived in the literature [2].

4 Conclusion

The theorem of Goldbach concerning the values of polynomials states that an infinite number of composite integers occur. By Dirichlet's theorem, there are an infinite number of primes in the arithmetic progression $an + b$ with a and b relatively prime. The generalization to polynomials would be the class of irreduducible polynomials over the integers such that the coefficients have no common prime divisor.

The proof of the existence of an infinite number of prime values of primitive irreducible polynomials follows from the analysis of the sum over a prime sieve. An upper bound of the form $\alpha_2 \frac{\xi}{ln\ \xi}$ for all values of $f(x) \leq \xi$. These inequalities may be refined to include a lower bound which has the same form $\alpha_1 \frac{\xi}{ln\ \xi}$. With both bounds, an estimate of $\mathcal{O}\left(\frac{\xi}{ln\ \xi}\right)$ exists. It is compatible with the estimate in the prime number theorem.

References

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