Generating function and combinatorial proofs of Elder’s theorem

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Abstract: We revisit Elder’s theorem on integer partitions, which is a generalization of Stanley’s theorem. Two new proofs are presented. The first proof is based on certain tilings of $1 \times \infty$ boards while the second one is a consequence of a more general identity we prove using generating functions.

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1 Introduction

We remember that an integer partition of a given positive integer $n$ is a set of positive integers whose sum equals $n$ and every integer in this set is called a part of the partition. In what follows,
we revisit Elder’s theorem, which relates the number of appearances of a given part among all partitions of $n$ with the number of repetitions of parts, i.e.,

**Theorem 1** (Elder’s theorem). *Let $n$ and $k$ be positive integers. The number of parts equal to $k$ in all partitions of $n$ is equal to the number of parts that appear at least $k$ times in a given partition of $n$, summed over all partitions of $n$. *

For example, considering the 7 partitions of 5, namely $5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1$, we see four parts 2 and the sum of the number of parts appearing at least twice in each of these partitions is $0 + 0 + 0 + 1 + 1 + 1 + 1 = 4$.

Elder’s theorem is a generalization of the following theorem, known as Stanley’s theorem:

**Theorem 2** (Stanley’s theorem). *Let $n$ be a positive integer. The number of parts equal to 1 in all partitions of $n$ is equal to the number of parts that appear at least once in a given partition of $n$, summed over all partitions of $n$. *

In 1982, M. S. Kirdar and T. H. R. Skyrme presented a generating function based proof of Elder’s theorem in [6]. A combinatorial proof of this identity can be found in [10], page 59, or in [8]. Some generalizations of both Elder’s and Stanley’s theorem are presented in [4].

In Section 2 we show a simple and elegant proof of Elder’s theorem using tilings. In Section 3, we use generating functions in order to obtain a proof of a partition identity from which Elder’s theorem follows as a particular case.

Our tiling proof of Elder’s theorem was inspired in recent applications of this technique to prove certain results (see, for example, [2, 3, 7, 9]). An introduction to this subject can be found in [1].

## 2 The first proof

In order to present our combinatorial proof of Elder’s theorem, we will need the following result.

**Lemma 3.** *Let $n$, $k$, and $r$ be positive integers. The number of partitions of $n$ with at least $k$ parts $r$ is equal to the number of partitions of $n$ having at least $r$ parts $k$. *

We prove this lemma via tilings. Consider a $1 \times \infty$ board that will be tiled using white and finitely many black squares, allowing stacking the black ones. All white tiles have weight 1 and the weight of a black tile in position $i$ is defined as $q^i$. Then, we define the weight of a tiling $T$ as the product of its tiles weights: $w(T) = \prod_{t \in T} w(t)$, where $w(t)$ is the weight of tile $t \in T$.

**Proof.** Let $X^r_k$ be the set of all tilings $T$ with at least $k$ black tiles in position $r$. We define the function

$$
\varphi^k_r : X^r_k \to X^k_r
$$

in the following way. Given $T \in X^r_k$, remove $k$ black squares from position $r$ and add $r$ black squares in position $k$, obtaining $T' = \varphi^k_r(T) \in X^k_r$. This function is clearly a bijection whose inverse is $\varphi^k_r$. It is also clear that $w(\varphi^k_r(T)) = w(T), \forall T \in X^r_k$. 

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In order to finish the proof we exhibit a bijection between these tilings of weight \( n \) and the partitions of \( n \). Let \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_j \) be a partition of \( n \). We associate to \( \lambda \) the tiling \( T_\lambda \) having a black square in each position \( \lambda_1, \ldots, \lambda_j \). Note that we may have more than 1 black square in a position since the parts of \( \lambda \) are not necessarily distinct.

Then \( w(T_\lambda) = q^{\lambda_1 + \lambda_2 + \cdots + \lambda_j} = q^n \).

In the proof of the theorem below we use the same tilings of the \( 1 \times \infty \) board described above.

**Theorem 4** (Elder’s theorem). Let \( n \) and \( k \) be positive integers. The number of parts equal to \( k \) in all partitions of \( n \) is equal to the number of parts that appear at least \( k \) times in a given partition of \( n \), summed over all partitions of \( n \).

**Proof.** Let \( Y^r_k \) be the subset of \( X^r_k \) consisting of the tilings with weight \( q^n \). According to the previous lemma, \( \varphi^r_k : X^r_k \to X^r_k \) is a weight preserving bijection for each \( r = 1, 2, \ldots \). Hence,

\[
\sum_{r=1}^{\infty} |Y^r_k| = \sum_{r=1}^{\infty} |Y^r_k|.
\]

The l.h.s. of (1) counts the numbers of tilings of weight \( q^n \) having \( k \) black squares in positions \( r = 1, 2, \ldots \), then this sum counts the number of parts that appear at least \( k \) times in the partitions of \( n \).

Each \( |Y^r_k| \) counts the number of tilings of weight \( q^n \) with at least \( r \) black squares in the position \( k \), which correspond to partitions of \( n \) with \( r \) parts equal to \( k \). It is easy to see that if \( T \in Y^r_s \), then \( T \in Y^s_r \), for \( s = 1, 2, \ldots, r \). Hence the r.h.s. of (1) counts the number of times \( k \) appears as part of the partitions of \( n \).

3 The second proof

We now prove a certain partition identity from which it follows a new proof of Elder’s theorem.

Let \( n \) and \( k_1, \ldots, k_r \) be positive integers with \( n \geq k_i \) and \( k_i \neq k_j \), if \( i \neq j \), \( P(n) \) the set of all partitions of \( n \), \( p(n) \) the number of partitions of \( n \) and \( \lambda \) a partition in \( P(n) \). We define \( f_{k_1, \ldots, k_r}(\lambda) \) as the number of times that the integers \( k_1, \ldots, k_r \) appear in \( \lambda \) and \( g_{k_1, \ldots, k_r}(\lambda) = \sum_{i=1}^{r} g_{k_i}(\lambda) \), where \( g_{k_i}(\lambda) \) is the number of parts in \( \lambda \) appearing at least \( k_i \) times. For example, considering \( \lambda = 5 + 4 + 4 + 3 + 2 + 2 + 2 + 1 \), a partition of \( n = 23 \), \( k_1 = 2 \), and \( k_2 = 3 \), we have:

\[
\begin{align*}
    f_{2,3}(5 + 4 + 4 + 3 + 2 + 2 + 2 + 1) &= 4, \\
    g_{2,3}(5 + 4 + 4 + 3 + 2 + 2 + 2 + 1) &= 3.
\end{align*}
\]

As another example, Table 1 below shows, for the eleven partitions of 6, a few values of \( f_{k_1, \ldots, k_r}(\lambda) \) as well as of \( g_{k_1, \ldots, k_r}(\lambda) \).

Our goal now is to prove the identity

\[
\sum_{\lambda \in P(n)} g_{k_1, \ldots, k_r}(\lambda) = \sum_{\lambda \in P(n)} f_{k_1, \ldots, k_r}(\lambda).
\]

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Partitions $\lambda$ of 6 | $f_{2,3}(\lambda)$ | $g_{2,3}(\lambda)$ | $f_{2,3,5}(\lambda)$ | $g_{2,3,5}(\lambda)$ |
---|---|---|---|---|
6    | 0  | 0  | 0  | 0  |
5 + 1 | 0  | 0  | 1  | 0  |
4 + 2 | 1  | 0  | 1  | 0  |
3 + 3 | 2  | 1  | 2  | 1  |
4 + 1 + 1 | 0  | 1  | 0  | 1  |
3 + 2 + 1 | 2  | 0  | 2  | 0  |
2 + 2 + 2 | 3  | 2  | 3  | 2  |
3 + 1 + 1 + 1 | 1  | 2  | 1  | 2  |
2 + 2 + 1 + 1 | 2  | 2  | 2  | 2  |
2 + 1 + 1 + 1 + 1 | 0  | 2  | 0  | 3  |
1 + 1 + 1 + 1 + 1 + 1 | 0  | 2  | 0  | 3  |
Sum | 12 | 12 | 13 | 13 |

Table 1: Some values for $n = 6$

As a consequence, it will follow a proof of the identity known as Elder’s theorem

$$\sum_{\lambda \in P(n)} g_k(\lambda) = \sum_{\lambda \in P(n)} f_k(\lambda),$$

where $n$ and $k$ are integers with $n \geq k > 0$, which was proved by Kirdar and Skyrme in [6] in a different manner.

### 3.1 The generating function for $\sum_{\lambda \in P(n)} f_{k_1,\ldots,k_r}(\lambda)$

Using the standard notation

$$(q; q)_{\infty} = \prod_{i=0}^{\infty} (1 - q^i),$$

we define

$$F(z, q) = \frac{\prod_{i=1}^{r} (1 - q^{k_i})}{(q; q)_{\infty}} \cdot \frac{1}{\prod_{i=1}^{r} (1 - z q^{k_i})} = \sum_{n \geq j \geq 0} P_{k_1,\ldots,k_r}(n, j) z^{j} q^{n},$$

where, clearly, $P_{k_1,\ldots,k_r}(n, j)$ is the number of partitions of $n$ having $j$ parts equal to $k_1, \ldots$, and $k_r$. Then, by definition, we have

$$\sum_{\lambda \in P(n)} f_{k_1,\ldots,k_r}(\lambda) = \sum_{j=1}^{n} j \cdot P_{k_1,\ldots,k_r}(n, j). \quad (3)$$

Considering that the generating function for the right hand side of (3) is obtained from $\frac{\partial F}{\partial z}(1, q)$ we have that the generating function for $\sum_{\lambda \in P(n)} f_{k_1,\ldots,k_r}(\lambda)$ is:
\[
\sum_{i=1}^{r} q^{k_i} \prod_{i \neq j=1}^{r} (1 - q^{k_j}) \over (q; q)_{\infty} \prod_{j=1}^{r} (1 - q^{k_j}).
\] (4)

In order to finish the proof of (2), we want to show that (4) also is the generating function for \( \sum_{\lambda \in P(n)} g_{k_1, \ldots, k_r}(\lambda) \).

### 3.2 The generating function for \( \sum_{\lambda \in P(n)} g_{k_1, \ldots, k_r}(\lambda) \)

As the factor \( 1/(1 - q^m) = \sum_{i=0}^{\infty} q^{im} \) is responsible for the number of times that \( m \) appears and we are interested in counting just the partitions where each part appears at least \( k_l \) times, \( l = 1, 2, \ldots, r \), we consider the following sum:

\[
\sum_{i=0}^{\infty} q^{im} - \sum_{i=k_l}^{\infty} q^{im} + z \sum_{i=k_l}^{\infty} q^{im} = \sum_{i=0}^{\infty} q^{im} - q^{mk_l} \sum_{i=0}^{\infty} q^{im} + zq^{mk_l} \sum_{i=0}^{\infty} q^{im} = \frac{1}{1 - q^m} - \frac{q^{mk_l}}{1 - q^m} + \frac{zq^{mk_l}}{1 - q^m}.
\]

Thus, we define

\[
G(z, q) = \frac{1}{(q; q)_{\infty}} \left( \prod_{i=1}^{\infty} (1 - q^{ik_1} + zq^{ik_1}) + \cdots + \prod_{i=1}^{\infty} (1 - q^{ik_r} + zq^{ik_r}) \right)
\]

\[
= \frac{1}{(q; q)_{\infty}} \sum_{l=1}^{r} \prod_{i=1}^{\infty} (1 - q^{ik_l} + zq^{ik_l})
\]

\[
= \sum_{n \geq j \geq 0} Q_{k_1, \ldots, k_r}(n, j) z^j q^n,
\]

where \( Q_{k_1, \ldots, k_r}(n, j) \) is equal to the sum, for \( l = 1, 2, \ldots, r \), of the number of partitions of \( n \) having \( j \) parts appearing at least \( k_l \) times. Then, by definition,

\[
\sum_{\lambda \in P(n)} g_{k_1, \ldots, k_r}(\lambda) = \sum_{j=1}^{n} j \cdot Q_{k_1, \ldots, k_r}(n, j).
\]

Knowing that the generating function for the right hand side of the equality above is obtained from \( \frac{\partial G}{\partial z}(1, q) \) we find it by calculating this derivative using the definition of partial derivative:

\[
\frac{\partial G}{\partial z}(1, q) = \lim_{h \to 0} \frac{G(1 + h, q) - G(1, q)}{h}
\]

\[
= \lim_{h \to 0} \left( \sum_{i=1}^{r} \prod_{i=1}^{\infty} (1 - q^{ik_l} + (1 + h)q^{ik_l}) - r \right)
\]

\[
= \lim_{h \to 0} \left( \sum_{i=1}^{r} \prod_{i=1}^{\infty} (1 + hq^{ik_l}) - r \right).
\]
Note that \( \prod_{i=1}^{\infty} (1 + hq^{ik_i}) = 1 + \sum_{i=1}^{\infty} hq^{ik_i} + M_i(h, q) \), where \( M_i(h, q) \) is a series in which the powers of \( h \) are greater than or equal to 2, i.e., \( M_i(h, q) = h^2 R_i(h, q) \). Then,

\[
\frac{\partial G}{\partial z}(1, q) = \lim_{h \to 0} \left( \frac{\sum_{i=1}^{r} (1 + \sum_{i=1}^{\infty} hq^{ik_i} + M_i(h, q)) - r}{h(q; q)_{\infty}} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{h(\sum_{i=1}^{\infty} q^{ik_1} + \cdots + \sum_{i=1}^{r} q^{ik_r}) + \sum_{i=1}^{r} M_i(h, q)}{h(q; q)_{\infty}} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{\sum_{i=1}^{r} q^{k_i} - \sum_{i=1}^{r} hR_i(h, q)}{h(q; q)_{\infty}} \right)
\]

\[
= \sum_{i=1}^{r} \frac{q^{k_i}}{(q; q)_{\infty}} - \sum_{i=1}^{r} \frac{q^{k_i}}{(q; q)_{\infty}} \prod_{i \neq j=1}^{r} (1 - q^{k_j})
\]

Clearly, as the generating functions for \( \sum_{\lambda \in P(n)} f_{k_1, \ldots, k_r}(\lambda) \) and for \( \sum_{\lambda \in P(n)} g_{k_1, \ldots, k_r}(\lambda) \) are the same, we have proved (2).

References


