Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 21, 2015, No. 4, 30–35

Generating function and combinatorial proofs of Elder's theorem

Robson da Silva[1], Jorge F. A. Lima[2], José Plínio O. Santos[3] and Eduardo C. Stabel[4]

[1] ICT, UNIFESP
12247-014, São José dos Campos-SP, Brazil
e-mail: silva.robson@unifesp.br

[2] IMECC, UNICAMP
C.P. 6065, 13084-970, Campinas-SP, Brazil
e-mail: jorge.fa.lima@gmail.com

[3] IMECC, UNICAMP
C.P. 6065, 13084-970, Campinas-SP, Brazil
e-mail: josepli@ime.unicamp.br

[4] UFSM
97105-900, Santa Maria-RS, Brazil
e-mail: dudastabel@gmail.com

Abstract: We revisit Elder's theorem on integer partitions, which is a generalization of Stanley's theorem. Two new proofs are presented. The first proof is based on certain tilings of $1 \times \infty$ boards while the second one is a consequence of a more general identity we prove using generating functions.
Keywords: Elder's theorem, Integer partition, Generating function, Tiling.
AMS Classification: 11P84, 05A19.

1 Introduction

We remember that an integer partition of a given positive integer n is a set of positive integers whose sum equals n and every integer in this set is called a part of the partition. In what follows, we revisit Elder's theorem, which relates the number of appearances of a given part among all partitions of n with the number of repetitions of parts, i.e.,

Theorem 1 (Elder's theorem)**.** *Let n and k be positive integers. The number of parts equal to k in all partitions of n is equal to the number of parts that appear at least k times in a given partition of n, summed over all partitions of n.*

For example, considering the 7 partitions of 5, namely $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$, we see four parts 2 and the sum of the number of parts appearing at least twice in each of these partitions is $0+0+0+1+1+1+1 = 4$.

Elder's theorem is a generalization of the following theorem, known as Stanley's theorem:

Theorem 2 (Stanley's theorem)**.** *Let n be a positive integer. The number of parts equal to 1 in all partitions of n is equal to the number of parts that appear at least once in a given partition of n, summed over all partitions of n.*

In 1982, M. S. Kirdar and T. H. R. Skyrme presented a generating function based proof of Elder's theorem in [6]. A combinatorial proof of this identity can be found in [10], page 59, or in [8]. Some generalizations of both Elder's and Stanley's theorem are presented in [4].

In Section 2 we show a simple and elegant proof of Elder's theorem using tilings. In Section 3, we use generating functions in order to obtain a proof of a partition identity from which Elder's theorem follows as a particular case.

Our tiling proof of Elder's theorem was inspired in recent applications of this technique to prove certain results (see, for example, [2, 3, 7, 9]). An introduction to this subject can be found in [1].

2 The first proof

In order to present our combinatorial proof of Elder's theorem, we will need the following result.

Lemma 3. *Let n, k, and r be positive integers. The number of partitions of n with at least k parts r is equal to the number of partitions of n having at least r parts k.*

We prove this lemma via tilings. Consider a $1 \times \infty$ board that will be tiled using white and finitely many black squares, allowing stacking the black ones. All white tiles have weight 1 and the weight of a black tile in position i is defined as q^i. Then, we define the weight of a tiling T as the product of its tiles weights: $w(T) = \prod_{t \in T} w(t)$, where $w(t)$ is the weight of tile $t \in T$.

Proof. Let X_k^r be the set of all tilings T with at least k black tiles in position r. We define the function

$$\varphi_k^r : X_k^r \to X_r^k$$

in the following way. Given $T \in X_k^r$, remove k black squares from position r and add r black squares in position k, obtaining $T' = \varphi_k^r(T) \in X_r^k$. This function is clearly a bijection whose inverse is φ_r^k. It is also clear that $w(\varphi_k^r(T)) = w(T), \forall T \in X_k^r$.

In order to finish the proof we exhibit a bijection between these tilings of weight n and the partitions of n. Let $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_j$ be a partition of n. We associate to λ the tilling T_λ having a black square in each position $\lambda_1, ..., \lambda_j$. Note that we may have more than 1 black square in a position since the parts of λ are not necessarily distinct.

Then $w(T_\lambda) = q^{\lambda_1+\lambda_2+\cdots+\lambda_j} = q^n$. □

In the proof of the theorem below we use the same tilings of the $1 \times \infty$ board described above.

Theorem 4 (Elder's theorem). *Let n and k be positive integers. The number of parts equal to k in all partitions of n is equal to the number of parts that appear at least k times in a given partition of n, summed over all partitions of n.*

Proof. Let Y_k^r be the subset of X_k^r consisting of the tilings with weight q^n. According to the previous lemma, $\varphi_k^r : X_k^r \to X_r^k$ is a weight preserving bijection for each $r = 1, 2, ...$ Hence,

$$\sum_{r=1}^{\infty} |Y_k^r| = \sum_{r=1}^{\infty} |Y_r^k|. \tag{1}$$

The l.h.s. of (1) counts the numbers of tilings of weight q^n having k black squares in positions $r = 1, 2, ...,$ then this sum counts the number of parts that appear at least k times in the partitions of n.

Each $|Y_r^k|$ counts the number of tilings of weight q^n with at least r black squares in the position k, which correspond to partitions of n with r parts equal to k. It is easy to see that if $T \in Y_r^k$, then $T \in Y_s^k$, for $s = 1, 2, ..., r$. Hence the r.h.s. of (1) counts the number of times k appears as part of the partitions of n. □

3 The second proof

We now prove a certain partition identity from which it follows a new proof of Elder's theorem.

Let n and $k_1, ..., k_r$ be positive integers with $n \geq k_i$ and $k_i \neq k_j$, if $i \neq j$, $P(n)$ the set of all partitions of n, $p(n)$ the number of partitions of n and λ a partition in $P(n)$. We define $f_{k_1,...,k_r}(\lambda)$ as the number of times that the integers $k_1, ..., k_r$ appear in λ and $g_{k_1,...,k_r}(\lambda) = \sum_{i=1}^{r} g_{k_i}(\lambda)$, where $g_{k_i}(\lambda)$ is the number of parts in λ appearing at least k_i times. For example, considering $\lambda = 5+4+4+3+2+2+2+1$, a partition of $n = 23$, $k_1 = 2$, and $k_2 = 3$, we have:

$$f_{2,3}(5+4+4+3+2+2+2+1) = 4,$$
$$g_{2,3}(5+4+4+3+2+2+2+1) = 3.$$

As another example, Table 1 below shows, for the eleven partitions of 6, a few values of $f_{k_1,...,k_r}(\lambda)$ as well as of $g_{k_1,...,k_r}(\lambda)$.

Our goal now is to prove the identity

$$\sum_{\lambda \in P(n)} g_{k_1,...,k_r}(\lambda) = \sum_{\lambda \in P(n)} f_{k_1,...,k_r}(\lambda). \tag{2}$$

Partitions λ of 6	$f_{2,3}(\lambda)$	$g_{2,3}(\lambda)$	$f_{2,3,5}(\lambda)$	$g_{2,3,5}(\lambda)$
6	0	0	0	0
$5+1$	0	0	1	0
$4+2$	1	0	1	0
$3+3$	2	1	2	1
$4+1+1$	0	1	0	1
$3+2+1$	2	0	2	0
$2+2+2$	3	2	3	2
$3+1+1+1$	1	2	1	2
$2+2+1+1$	2	2	2	2
$2+1+1+1+1$	1	2	1	2
$1+1+1+1+1+1$	0	2	0	3
Sum	12	12	13	13

Table 1: Some values for $n = 6$

As a consequence, it will follow a proof of the identity known as Elder's theorem

$$\sum_{\lambda \in P(n)} g_k(\lambda) = \sum_{\lambda \in P(n)} f_k(\lambda),$$

where n and k are integers with $n \geq k > 0$, which was proved by Kirdar and Skyrme in [6] in a different manner.

3.1 The generating function for $\sum_{\lambda \in P(n)} f_{k_1,\ldots,k_r}(\lambda)$

Using the standard notation

$$(q;q)_\infty = \prod_{i=0}^{\infty}(1-q^i),$$

we define

$$F(z,q) = \frac{\prod_{i=1}^{r}(1-q^{k_i})}{(q;q)_\infty} \cdot \frac{1}{\prod_{i=1}^{r}(1-zq^{k_i})} = \sum_{n \geq j \geq 0} P_{k_1,\ldots,k_r}(n,j) z^j q^n,$$

where, clearly, $P_{k_1,\ldots,k_r}(n,j)$ is the number of partitions of n having j parts equal to k_1, ..., and k_r. Then, by definition, we have

$$\sum_{\lambda \in P(n)} f_{k_1,\ldots,k_r}(\lambda) = \sum_{j=1}^{n} j \cdot P_{k_1,\ldots,k_r}(n,j). \tag{3}$$

Considering that the generating function for the right hand side of (3) is obtained from $\frac{\partial F}{\partial z}(1,q)$ we have that the generating function for $\sum_{\lambda \in P(n)} f_{k_1,\ldots,k_r}(\lambda)$ is:

$$\frac{\sum_{i=1}^{r} q^{k_i} \prod_{i\neq j=1}^{r} (1-q^{k_j})}{(q;q)_\infty \prod_{j=1}^{r}(1-q^{k_j})}. \tag{4}$$

In order to finish the proof of (2), we want to show that (4) also is the generating function for $\sum_{\lambda\in P(n)} g_{k_1,\dots,k_r}(\lambda)$.

3.2 The generating function for $\sum_{\lambda\in P(n)} g_{k_1,\dots,k_r}(\lambda)$

As the factor $1/(1-q^m) = \sum_{i=0}^{\infty} q^{im}$ is responsible for the number of times that m appears and we are interested in counting just the partitions where each part appears at least k_l times, $l = 1, 2, ..., r$, we consider the following sum:

$$\begin{aligned}\sum_{i=0}^{\infty} q^{im} - \sum_{i=k_l}^{\infty} q^{im} + z\sum_{i=k_l}^{\infty} q^{im} &= \sum_{i=0}^{\infty} q^{im} - q^{mk_l}\sum_{i=0}^{\infty} q^{im} + zq^{mk_l}\sum_{i=0}^{\infty} q^{im}\\ &= \frac{1}{(1-q^m)} - \frac{q^{mk_l}}{(1-q^m)} + \frac{zq^{mk_l}}{(1-q^m)}.\end{aligned}$$

Thus, we define

$$\begin{aligned}G(z,q) &= \frac{1}{(q;q)_\infty}\left(\prod_{i=1}^{\infty}(1-q^{ik_1}+zq^{ik_1}) + \cdots + \prod_{i=1}^{\infty}(1-q^{ik_r}+zq^{ik_r})\right)\\ &= \frac{1}{(q;q)_\infty}\sum_{l=1}^{r}\prod_{i=1}^{\infty}(1-q^{ik_l}+zq^{ik_l})\\ &= \sum_{n\geq j\geq 0} Q_{k_1,\dots,k_r}(n,j) z^j q^n,\end{aligned}$$

where $Q_{k_1,\dots,k_r}(n,j)$ is equal to the sum, for $l = 1, 2, ..., r$, of the number of partitions of n having j parts appearing at least k_l times. Then, by definition,

$$\sum_{\lambda\in P(n)} g_{k_1,\dots,k_r}(\lambda) = \sum_{j=1}^{n} j\cdot Q_{k_1,\dots,k_r}(n,j).$$

Knowing that the generating function for the right hand side of the equality above is obtained from $\frac{\partial G}{\partial z}(1,q)$ we find it by calculating this derivative using the definition of partial derivative:

$$\begin{aligned}\frac{\partial G}{\partial z}(1,q) &= \lim_{h\to 0}\frac{G(1+h,q)-G(1,q)}{h}\\ &= \lim_{h\to 0}\left(\frac{\sum_{l=1}^{r}\prod_{i=1}^{\infty}(1-q^{ik_l}+(1+h)q^{ik_l})-r}{h(q;q)_\infty}\right)\\ &= \lim_{h\to 0}\left(\frac{\sum_{l=1}^{r}\prod_{i=1}^{\infty}(1+hq^{ik_l})-r}{h(q;q)_\infty}\right).\end{aligned}$$

Note that $\prod_{i=1}^{\infty}(1+hq^{ik_l}) = 1+\sum_{i=1}^{\infty} hq^{ik_l} + M_l(h,q)$, where $M_l(h,q)$ is a series in which the powers of h are greater than or equal to 2, i.e., $M_l(h,q) = h^2 R_l(h,q)$. Then,

$$
\begin{aligned}
\frac{\partial G}{\partial z}(1,q) &= \lim_{h\to 0}\left(\frac{\sum_{l=1}^{r}(1+\sum_{i=1}^{\infty} hq^{ik_l} + M_l(h,q)) - r}{h(q;q)_\infty}\right)\\
&= \lim_{h\to 0}\left(\frac{h(\sum_{i=1}^{\infty} q^{ik_1} + \cdots + \sum_{i=1}^{\infty} q^{ik_r}) + \sum_{l=1}^{r} M_l(h,q)}{h(q;q)_\infty}\right)\\
&= \lim_{h\to 0}\left(\sum_{i=1}^{r}\frac{q^{k_i}}{(1-q^{k_i})(q;q)_\infty} + \frac{\sum_{l=1}^{r} hR_l(h,q)}{(q;q)_\infty}\right)\\
&= \sum_{i=1}^{r}\frac{q^{k_i}}{(1-q^{k_i})(q;q)_\infty}\\
&= \frac{\sum_{i=1}^{r} q^{k_i}\prod_{i\neq j=1}^{r}(1-q^{k_j})}{(q;q)_\infty \prod_{j=1}^{r}(1-q^{k_j})}.
\end{aligned}
$$

Clearly, as the generating functions for $\sum_{\lambda\in P(n)} f_{k_1,\ldots,k_r}(\lambda)$ and for $\sum_{\lambda\in P(n)} g_{k_1,\ldots,k_r}(\lambda)$ are the same, we have proved (2).

References

[1] Benjamin, A. T., & J. J. Quinn (2003) Proofs That Really Count: The Art of Combinatorial Proof. *The Dolciani Mathematical Expositions*, 27.

[2] Benjamin, A. T., Plott, S., & J. A. Sellers (2008) Tiling Proofs of Recent Sum Identities Involving Pell Numbers. *Annals of Combinatorics*, 12, 271–278.

[3] Briggs, K. S., Little, D. P., & J. A. Sellers (2011) Combinatorial Proofs of Various q-Pell Identities via Tilings. *Annals of Combinatorics*, 14(4), 407–418.

[4] Dastidar, M. G., & S. S. Gupta (2013) Generalization of a few results in integer partitions. *Notes on Number Theory and Discrete Mathematics*, 19(2), 69–76.

[5] Hoare, A. H. (1986) An Involution of Blocks in the Partitions of n, *The American Mathematical Monthly*, 93, 475–476.

[6] Kirdar, M. S., & T. H. R. Skyrme (1982) On an Identity Related to Partitions and Repetitions of Parts, *Canadian Journal of Mathematics*, 34, 194–195.

[7] Little, D. P., & J. A. Sellers (2010) A Tiling Approach to Eight Identities of Rogers. *European Journal of Combinatorics*, 31, 694–709.

[8] Schmidt, F. W., & R. Simion (1984) On a Partition Identity, *Journal of Combinatorial Theory Series A*, Vol. 36, 249–252.

[9] Stabel, E. C. (2011) A Combinatorial Proof of an Identity of Ramanujan Using Tilings. *Bulletin of the Brazilian Mathematical Society*, 42(2), 203–212.

[10] Stanley, R. P. (1997) Enumerative Combinatorics, Vol. 1, Cambridge University Press, UK.