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Some Diophantine equations concerning biquadrates

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Abstract: This paper is concerned with integer solutions of the diophantine equation $x_1^4 + x_2^4 + x_3^4 = kx_4^2$ where k is a given positive integer. Till now, integer and parametric solutions of this diophantine equation have been published only when $k = 1$ or 2 or 3. In this paper we obtain parametric solutions of this equation for 43 values of $k \leqslant 100$. We also show that the equation cannot have any solution in integers for 54 values of $k \leqslant 100$. The solvability of the equation $x_1^4 + x_2^4 + x_3^4 = kx_4^2$ could not be determined for three values of $k \leqslant 100$, namely $34,\ 35$ and 65.
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1 Introduction

This paper is concerned with integer solutions of the diophantine equation,

$$x_1^4 + x_2^4 + x_3^4 = kx_4^2, \tag{1}$$

where k is a given positive integer. Diophantus [2, p. 224] gave an elegant solution in rational numbers of Eq. (1) when $k = 1$. The method of Diophantus leads to a parametric solution [1, p. 657] that readily yields integer solutions of Eq. (1) when $k = 1$. Proth [4] has given a parametric solution of Eq. (1) when $k = 2$, while Realis has given a parametric solution of Eq. (1) when $k = 2$ [6], and also when $k = 3$ [5]. No solutions of Eq. (1) have been published for any value of $k > 3$. In fact, Piezas [3] asks whether there are any parametric solutions of Eq. (1) when $k > 3$.

In this paper we obtain numerical and parametric solutions of Eq. (1) for 43 values of $k \leqslant 100$. We also show that integer solutions of (1) cannot exist for 54 values of $k \leqslant 100$. The solvability of (1) for three values of $k \leqslant 100$, namely $34,\ 35$ and 65 could not be determined.

In Section 2 we give some preliminary lemmas concerning the solvability of Eq. (1). In Section 3 we obtain numerical and parametric solutions of (1) for various values of k while in Section 4 we mention some open problems related to Eq. (1).

2 Preliminary lemmas

Eq. (1) has the trivial solution $(x_1,\, x_2,\, x_3,\, x_4) = (0,\, 0,\, 0,\, 0)$ for all integer values of k. Accordingly, when we refer to a solution of Eq. (1), we will mean a nontrivial solution, that is, a solution in which $x_i,\ i = 1,\, 2,\, 3,\, 4$, are not all zero. We note that if $(x_1,\, x_2,\, x_3,\, x_4) = (\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4)$ is a solution of Eq. (1), then for any arbitrary value of h, $(x_1,\, x_2,\, x_3,\, x_4) = (h\alpha_1,\, h\alpha_2,\, h\alpha_3,\, h^2\alpha_4)$ is also a solution of (1). We will now prove certain preliminary lemmas concerning Eq. (1).

Lemma 1. If k and m are integers, the equation

$$x_1^4 + x_2^4 + x_3^4 = km^2x_4^2, \tag{2}$$

has a solution in integers if and only if Eq. (1) has a solution in integers.
Proof: If $(x_1,\, x_2,\, x_3,\, x_4) = (\beta_1,\, \beta_2,\, \beta_3,\, \beta_4)$ is a solution of Eq. (1), then $(x_1,\, x_2,\, x_3,\, x_4) = (\beta_1 m,\, \beta_2 m,\, \beta_3 m,\, \beta_4 m)$ is a solution of (2). Conversely, if $(x_1,\, x_2,\, x_3,\, x_4) = (\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4)$ is a solution of (2), then $(x_1,\, x_2,\, x_3,\, x_4) = (\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4 m)$ is a solution of (1). This proves the lemma. □

Lemma 2. Eq. (1) has no solution in integers if $k \equiv 5,\, 6,\, 7,\, 10,\, 13,\, 14,\, 15 \pmod{16}$.
Proof: If $(x_1,\, x_2,\, x_3,\, x_4) = (\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4)$ is a solution in integers of (1), we may write $\alpha_i = 2^{e_i}\beta_i,\ i = 1,\, 2,\, 3$, where $\beta_1,\, \beta_2,\, \beta_3$ are odd integers, and without loss of generality, we may take e_1 to be the least of the integers $e_1,\, e_2,\, e_3$ so that $2^{4e_1} \mid k\alpha_4^2$. If $k \equiv 5,\, 6,\, 7,\, 10,\, 13,\, 14,\, 15$ $\pmod{16}$, then $2^2 \nmid k$, and so we must have $2^{4e_1-1} \mid \alpha_4^2$. It follows that $2^{2e_1} \mid \alpha_4$, and hence, from the initial solution of (1), we derive another integer solution of (1), namely $(x_1,\, x_2,\, x_3,\, x_4) = (\beta_1,\, 2^{e_2-e_1}\beta_2,\, 2^{e_3-e_1}\beta_3,\, \alpha_4/2^{2e_1})$. We have thus obtained a solution in integers of (1) in which at least one of the x_i on the lefthand side is odd. For any integer x, we have $x^4 \equiv 0,\ 1 \pmod{16}$, and so the lefthand side of (1) is $1,\, 2,$ or $3 \pmod{16}$. Further, for any integer x, we have $x^2 \equiv 0,\, 1,\, 4,\, 9 \pmod{16}$, and when $k \equiv 5,\, 6,\, 7,\, 10,\, 13,\, 14,\, 15 \pmod{16}$, it is easily seen that the righthand side of (1) cannot be $1,\, 2,$ or $3 \pmod{16}$. It follows that Eq. (1) cannot have any solution in integers when $k \equiv 5,\, 6,\, 7,\, 10,\, 13,\, 14,\, 15 \pmod{16}$. This proves the lemma. □

Lemma 3. Eq. (1) has no solution in integers for the following 54 values of $k \leqslant 100$:

$$\begin{gathered}5,\, 6,\, 7,\, 10,\, 13,\, 14,\, 15,\, 20,\, 21,\, 22,\, 23,\, 24,\, 26,\, 28,\, 29,\, 30,\, 31,\, 37,\, 38,\\ 39,\, 40,\, 42,\, 45,\, 46,\, 47,\, 52,\, 53,\, 54,\, 55,\, 56,\, 58,\, 60,\, 61,\, 62,\, 63,\, 69,\, 70,\\ 71,\, 74,\, 77,\, 78,\, 79,\, 80,\, 84,\, 85,\, 86,\, 87,\, 88,\, 90,\, 92,\, 93,\, 94,\, 95,\, 96.\end{gathered}$$

Proof: It immediately follows from Lemma 2, there are no integer solutions of Eq. (1) for 42 values of k listed here. Further, in view of Lemma 1, there cannot be integer solutions of Eq. (1) when each of these 42 values is multiplied by a perfect square, eg. 4 or 16. This rules out solutions of (1) when $k = 20, 24, 28, 40, 52, 56, 60, 80, 84, 88, 92$ and 96. This completes the proof. □

3 Numerical and parametric solutions of Equation (1)

We obtained integer solutions of (1) by conducting trials over the range $x_1 + x_2 + x_3 \leqslant 5600$. This yielded integer solutions of (1) for 20 square-free integer values of $k \leqslant 100$. The smallest solution in positive integers of Eq. (1) for each such value of k is listed in Table 3. Further, by applying Lemma 1 and using the solutions in Table 3, we readily obtain solutions in positive integers of (1) for the following additional 23 values of k in which k contains a squared factor:

$$4,\ 8,\ 9,\ 12,\ 16,\ 18,\ 25,\ 27,\ 32,\ 36,\ 44,\ 48,\ 49,\ 50,\ 64,\ 68,\ 72,\ 75,\ 76,\ 81,\ 98,\ 99,\ 100.$$

We thus get positive integer solutions of (1) for 43 values of k less than 100.

We note that for certain values of k, Eq. (1) cannot have a solution in coprime integers. For instance when k is divisible by 4 but not by 16, then all the integers $x_i,\ i = 1, 2, 3, 4$ must be even. If any of the three integers $x_i,\ i = 1, 2, 3$ is odd, as seen in the proof of Lemma 2, the lefthand side of (1) is $1, 2$, or $3 \pmod{16}$, hence not divisible by 4. Thus, $x_i,\ i = 1, 2, 3$ must all be even integers and now it readily follows that x_4 must also be an even integer.

We now show that if one integer solution of (1) is known, we can find a parametric solution of (1). Let $(x_1,\ x_2,\ x_3,\ x_4) = (a_1,\ a_2,\ a_3,\ a_4)$ be a known solution of (1) so that $a_1^4 + a_2^4 + a_3^4 = ka_4^2$. We now write,

$$x_1 = a_1t_1 + ka_4pt_2,\quad x_2 = a_2t_1 + ka_4qt_2,\quad x_3 = a_3t_1,\quad x_4 = a_4t_1^2 + rt_1t_2 + st_2^2, \tag{3}$$

where $p,\ q,\ r,\ s,\ t_1$ and t_2 are arbitrary parameters. On substituting these values in (1) and transposing all terms to the lefthand side, we note that the coefficient of t_1^4 vanishes and (1) reduces to the following cubic equation:

$$\begin{aligned}2a_4(2a_1^3p + 2a_2^3q - r)t_1^3 &+ (6ka_1^2a_4^2p^2 + 6ka_2^2a_4^2q^2 - 2a_4s - r^2)t_1^2t_2 \\ &+ 2(2k^2a_1a_4^3p^3 + 2k^2a_2a_4^3q^3 - rs)t_1t_2^2 + (k^3a_4^4p^4 + k^3a_4^4q^4 - s^2)t_2^3 = 0.\end{aligned} \tag{4}$$

We equate the coefficients of t_1^3 and $t_1^2t_2$ in (4) to 0, and solve these equations for r and s to get the following solution:

$$\begin{aligned}r &= 2a_1^3p + 2a_2^3q,\\ s &= -\{(2a_1^4 - 3ka_4^2)a_1^2p^2 + 4a_1^3a_2^3pq + (2a_2^4 - 3ka_4^2)a_2^2q^2\}/a_4.\end{aligned} \tag{5}$$

With these values of $r,\ s$, Eq. (4) reduces to a linear equation in t_1 and t_2 and its solution may be written as follows:

$$\begin{aligned}t_1 &= (k^3a_4^4p^4 + k^3a_4^4q^4 - s^2)a_4^2,\\ t_2 &= -2(2k^2a_1a_4^3p^3 + 2k^2a_2a_4^3q^3 - rs)a_4^2.\end{aligned} \tag{6}$$

k	x_1	x_2	x_3	x_4
1	12	15	20	481
2	1	1	2	3
3	1	1	1	1
11	3	5	5	11
17	6	17	22	137
19	1	15	15	73
33	1	2	2	1
41	275	292	820	106509
43	3	5	9	13
51	425	563	565	67881
57	1	4	4	3
59	5	45	97	1253
66	5	5	8	9
67	5	7	7	9
73	2	2	5	3
82	23	28	41	213
83	1	1	3	1
89	29	40	60	427
91	45	51	55	469
97	33	54	190	3679

Table 1. Solutions of $x_1^4 + x_2^4 + x_3^4 = kx_4^2$

It follows that when the values of r and s are given by (5) and the values of t_1 and t_2 are given by (6), a solution in integers of Eq. (1) is given by (3) in terms of arbitrary integer parameters p and q. The factor a_4^2 in the values of t_1 and t_2 ensures that both t_1 and t_2 are integers for integer values of p and q even when the value of s given by (5) is not an integer. Thus the parametric solution (3) yields integer solutions of Eq. (1) for all integer values of the parameters p and q.

As an example, when $k = 11$, referring to Table 3 we take $(a_1, a_2, a_3, a_4) = (3, 5, 5, 11)$ as the initial known solution, and following the above method, we obtain a solution of the equation $x_1^4 + x_2^4 + x_3^4 = 11x_4^2$ in terms of parameters p and q. In this solution we replace p by $p/5$, and denoting the polynomial $c_0p^n + c_1p^{n-1}q + c_2p^{n-2}q^2 + \cdots + c_nq^n$ by $(c_0, c_1, c_2, \ldots, c_n)$, this two-parameter solution may be written as follows:

$$x_1 = (-31730634, 190383804, -554394618, 806456736, -7033748802),$$
$$x_2 = (9353170, -149475360, -142001910, 2026178820, -13245205990),$$
$$x_3 = (9353170, 37237320, -982208970, 1851525000, -11722914670),$$

$$x_4 = (305854850695724, -3670258208348688, 22352428383995160, - 85558721892972768, 201095678922153092, -270032645827278048, 7188041983481442072, -21058085417462341776, 68828742151520431916).$$

As we have obtained integer solutions of (1) for 43 values of $k \leqslant 100$, we readily get two-parameter solutions of (1) for these 43 values of k.

For any specific value of k, we may obtain the parametric solution (3) and then use it as the initial known solution and thus obtain a four-parameter solution of (1). This solution is, however, too cumbersome to be written explicitly.

4 Some open problems

As solutions of (1) could not be obtained when $k = 34,\ 35$ or 65, it would be of interest to obtain integer solutions of (1) for these values of k. It would be even more interesting to establish whether there exist integer solutions of (1) for all values of k except those values that have to be excluded because of Lemma 2 or there are other values of k for which Eq. (1) has no solutions in integers.

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