On the density of ranges of generalized divisor functions

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Abstract: The range of the divisor function \( \sigma_{-1} \) is dense in the interval \([1, \infty)\). However, although the range of the function \( \sigma_{-2} \) is a subset of the interval \([1, \frac{\pi^2}{6})\), we will see that the range of \( \sigma_{-2} \) is not dense in \([1, \frac{\pi^2}{6})\). We begin by generalizing the divisor functions to a class of functions \( \sigma_t \) for all real \( t \). We then define a constant \( \eta \approx 1.8877909 \) and show that if \( r \in (1, \infty) \), then the range of the function \( \sigma_{-r} \) is dense in the interval \([1, \zeta(r))\) if and only if \( r \leq \eta \). We end with an open problem.

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1 Introduction

Throughout this paper, we will let \( \mathbb{N} \) denote the set of positive integers, and we will let \( p_i \) denote the \( i^{th} \) prime number.

For any integer \( t \), the divisor function \( \sigma_t \) is a multiplicative arithmetic function defined by \( \sigma_t(n) = \sum_{d|n} d^t \) for all positive integers \( n \). The value of \( \sigma_1(n) \) is the sum of the positive divisors of \( n \), while the value of \( \sigma_0(n) \) is simply the number of positive divisors of \( n \).
Another interesting divisor function is $\sigma_{-1}$, which is often known as the abundancy index. One may show [2] that the range of $\sigma_{-1}$ is a subset of the interval $[1, \infty)$ that is dense in $[1, \infty)$. If $t < -1$, then the range of $\sigma_t$ is a subset of the interval $[1, \zeta(-t)]$, where $\zeta$ denotes the Riemann zeta function. This is because, for any positive integer $n$, $\sigma_t(n) = \sum_{d|n} d^t < \sum_{i=1}^{\infty} i^t = \zeta(-t)$. For example, the range of the function $\sigma_{-2}$ is a subset of the interval $[1, \pi^2/6)$. However, it is interesting to note that the range of the function $\sigma_{-2}$ is not dense in the interval $[1, \pi^2/6)$. To see this, let $n$ be a positive integer. If $2|n$, then $\sigma_{-2}(n) \geq \frac{1}{2} + \frac{1}{2^2} = \frac{5}{4}$. On the other hand, if $2 \nmid n$, then $\sigma_{-2}(n) < \sum_{d \in \mathbb{N} \setminus (2\mathbb{N})} \frac{1}{d^{2^t}} = \frac{\zeta(2)}{\left(\frac{1}{1-2^t}\right)} = \frac{\pi^2}{8}$. As $\pi^2/8 < 5/4$, we see that there is a “gap” in the range of $\sigma_{-2}$. In other words, there are no positive integers $n$ such that $\sigma_{-2}(n) \in \left(\frac{\pi^2}{8}, \frac{5}{4}\right)$.

Our first goal is to generalize the divisor functions to allow for nonintegral subscripts. For example, we might consider the function $\sigma_{\sqrt{2}}$, defined by $\sigma_{\sqrt{2}}(n) = \sum_{d|n} d^{\sqrt{2}}$. We formalize this idea in the following definition.

**Definition 1.1.** For a real number $t$, define the function $\sigma_t: \mathbb{N} \to \mathbb{R}$ by $\sigma_t(n) = \sum_{d|n} d^t$ for all $n \in \mathbb{N}$. Also, we will let $\log \sigma_t = \log \circ \sigma_t$.

In analyzing the ranges of these generalized divisor functions, we will find a constant which serves as a “boundary” between divisor functions with dense ranges and divisor functions with ranges that have gaps. Note that, for any real number $t$, we may write $\sigma_t = I_0 * I_t$, where $I_0$ and $I_t$ are arithmetic functions defined by $I_0(n) = 1$ and $I_t(n) = n^t$. As $I_0$ and $I_t$ are multiplicative, we find that $\sigma_t$ is multiplicative.

### 2 The ranges of functions $\sigma_{-r}$

**Theorem 2.1.** Let $r$ be a real number greater than 1. The range of $\sigma_{-r}$ is dense in the interval $[1, \zeta(r))$ if and only if $1 + \frac{1}{p_m} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^r}\right)$ for all positive integers $m$.

**Proof.** First, suppose that $1 + \frac{1}{p_m} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^r}\right)$ for all positive integers $m$. We will show that the range of $\log \sigma_{-r}$ is dense in the interval $[0, \log(\zeta(r))]$, which will imply that the range of $\sigma_{-r}$ is dense in $[1, \zeta(r))$. Choose some arbitrary $x \in (0, \log(\zeta(r)))$, and define $X_0 = 0$. For each
positive integer \( n \), we define \( \alpha_n \) and \( X_n \) in the following manner. If \( X_{n-1} + \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) \leq x \), define \( \alpha_n = -1 \). If \( X_{n-1} + \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) > x \), define \( \alpha_n \) to be the largest nonnegative integer that satisfies \( X_{n-1} + \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) \leq x \). Define \( X_n \) by

\[
X_n = \begin{cases} 
X_{n-1} + \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right), & \text{if } \alpha_n \geq 0; \\
X_{n-1} + \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right), & \text{if } \alpha_n = -1.
\end{cases}
\]

Also, for each \( n \in \mathbb{N} \), define \( D_n \) by

\[
D_n = \begin{cases} 
\log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) - \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right), & \text{if } \alpha_n \geq 0; \\
0, & \text{if } \alpha_n = -1,
\end{cases}
\]

and let \( E_n = \sum_{i=1}^{n} D_i \). Note that

\[
\lim_{n \to \infty} (X_n + E_n) = \lim_{n \to \infty} \left( X_n + \sum_{i=1}^{n} D_i \right) = \lim_{n \to \infty} \sum_{i=1}^{n} \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) = \log(\zeta(r)).
\]

Now, because the sequence \( (X_n)_{n=1}^{\infty} \) is bounded and monotonic, we know that there exists some real number \( \gamma \) such that \( \lim_{n \to \infty} X_n = \gamma \). We wish to show that \( \gamma = x \).

Notice that we defined the sequence \( (X_n)_{n=1}^{\infty} \) so that \( X_n \leq x \) for all \( n \in \mathbb{N} \). Hence, we know that \( \gamma \leq x \). Now, suppose \( \gamma < x \). Then \( \lim_{n \to \infty} E_n = \log(\zeta(r)) - \gamma > \log(\zeta(r)) - x \). This implies that there exists some positive integer \( N \) such that \( E_n > \log(\zeta(r)) - x \) for all integers \( n \geq N \). Let \( m \) be the smallest positive integer that satisfies \( E_m > \log(\zeta(r)) - x \). If \( \alpha_m = -1 \) and \( m > 1 \), then \( D_m = 0 \), so \( E_{m-1} = E_m > \log(\zeta(r)) - x \). However, this contradicts the minimality of \( m \). If \( \alpha_m = -1 \) and \( m = 1 \), then \( 0 = D_m = E_m > \log(\zeta(r)) - x \), which is also a contradiction. Thus, we conclude that \( \alpha_m \geq 0 \). This means that \( X_m + D_m = X_{m-1} + \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) > x \), so \( D_m > x - X_m \). Furthermore,

\[
\log \left( \prod_{i=m+1}^{\infty} \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) \right) = \sum_{i=m+1}^{\infty} \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) = \log(\zeta(r)) - \sum_{i=1}^{m} \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^n} \right) \]

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\[ \log(\zeta(r)) - E_m - X_m < x - X_m < D_m, \] (1)

and we originally assumed that \( 1 + \frac{1}{p_m} \leq \prod_{i=m+1}^{\infty} \left( \sum_{j=0}^{\infty} \frac{1}{p_i^{p_j}} \right). \) This means that

\[
\log \left( 1 + \frac{1}{p_m^r} \right) < D_m = \log \left( \sum_{j=0}^{\infty} \frac{1}{p_m^{r_j}} \right) - \log \left( \sum_{j=0}^{\alpha_m} \frac{1}{p_m^{r_j}} \right), \]

or equivalently,

\[
\log \left( 1 + \frac{1}{p_m^r} \right) + \log \left( \sum_{j=0}^{\alpha_m} \frac{1}{p_m^{r_j}} \right) < \log \left( \frac{p_m^r}{p_m^{r-1}} \right). \]

If \( \alpha_m > 0, \) we have

\[
\log \left( 1 + \frac{1}{p_m^r} \right) \leq \log \left( 1 + \frac{1}{p_m^r} \right) + \log \left( \sum_{j=0}^{\alpha_m} \frac{1}{p_m^{r_j}} \right) < \log \left( \frac{p_m^r}{p_m^{r-1}} \right),
\]

so \( \left( 1 + \frac{1}{p_m^r} \right)^2 < \frac{p_m^r}{p_m^{r-1}}. \) We may write this as \( 1 + \frac{2}{p_m^r} + \frac{1}{p_m^{r-1}} < 1 + \frac{1}{p_m^{r-1}} \), so

\[
2 \times \frac{p_m^r}{p_m^{r-1}} = 1 + \frac{1}{p_m^{r-1}}. \]

As \( p_m^r > 2, \) this is a contradiction. Hence, \( \alpha_m = 0. \) By the definitions of \( \alpha_m \) and \( X_m, \) this implies that \( X_{m-1} + \log \left( 1 + \frac{1}{p_m^r} \right) > x \) and that \( X_m = X_{m-1}. \)

Therefore, \( \log \left( 1 + \frac{1}{p_m^r} \right) > x - X_{m-1} = x - X_m. \) However, recalling from (1) that

\[
\sum_{i=m+1}^{\infty} \log \left( \sum_{j=0}^{\infty} \frac{1}{p_i^{r_j}} \right) < x - X_m,
\]

we find that

\[
\sum_{i=m+1}^{\infty} \log \left( \sum_{j=0}^{\infty} \frac{1}{p_i^{r_j}} \right) < \log \left( 1 + \frac{1}{p_m^r} \right),
\]

which is a contradiction because we originally assumed that \( 1 + \frac{1}{p_m^r} \leq \prod_{i=m+1}^{\infty} \left( \sum_{j=0}^{\infty} \frac{1}{p_i^{r_j}} \right). \) Therefore, \( \gamma = x. \)

We now know that \( \lim_{n \to \infty} X_n = x. \) To show that the range of \( \log \sigma_r \) is dense in \( [0, \log(\zeta(r))] \), we need to construct a sequence \( (C_n)_{n=1}^{\infty} \) of elements of the range of \( \log \sigma_r \) that satisfies \( \lim_{n \to \infty} C_n = x. \) We do so in the following fashion. For each positive integer \( n, \) write

\[
Y_n = \begin{cases} 
1, & \text{if } \alpha_n \geq 0; \\
0, & \text{if } \alpha_n = -1,
\end{cases}
\]

\[
Z_n = \begin{cases} 
0, & \text{if } \alpha_n \geq 0; \\
1, & \text{if } \alpha_n = -1,
\end{cases}
\]

and

\[
\beta_n = \begin{cases} 
\alpha_n, & \text{if } \alpha_n \geq 0; \\
0, & \text{if } \alpha_n = -1.
\end{cases}
\]
Now, for each positive integer $n$, define $C_n$ by

$$C_n = \sum_{k=1}^{n} \left( Y_k \log \left( \sum_{j=0}^{\beta_k} \frac{1}{p_j^{n}} \right) + Z_k \log \left( \sum_{j=0}^{n} \frac{1}{p_j^{n}} \right) \right).$$

Notice that, by the way we defined $X_n$, we have

$$X_n = \sum_{k=1}^{n} \left( Y_k \log \left( \sum_{j=0}^{\beta_k} \frac{1}{p_j^{n}} \right) + Z_k \log \left( \sum_{j=0}^{\infty} \frac{1}{p_j^{n}} \right) \right).$$

Therefore, $\lim_{n \to \infty} C_n = \lim_{n \to \infty} X_n = x$. All we need to do now is show that each $C_n$ is in the range of $\log \sigma_r$. We have

$$C_n = \sum_{k=1}^{n} \left( Y_k \log \left( \sum_{j=0}^{\beta_k} \frac{1}{p_j^{n}} \right) + Z_k \log \left( \sum_{j=0}^{n} \frac{1}{p_j^{n}} \right) \right)$$

$$= \sum_{k \in \mathbb{N}} \log \left( \sum_{j=0}^{\beta_k} \frac{1}{p_j^{n}} \right) + \sum_{k \in \mathbb{N}} \log \left( \sum_{j=0}^{n} \frac{1}{p_j^{n}} \right)$$

$$= \log \left( \prod_{k \in \mathbb{N}} \sigma_r(p_j^{\alpha_k}) \right) \left( \prod_{k \in \mathbb{N}} \sigma_r(p_j^{n}) \right)$$

$$= \log \sigma_r \left( \left( \prod_{k \leq n} p_j^{\alpha_k} \right) \left( \prod_{k \leq n} p_j^{n} \right) \right).$$

We finally conclude that if $1 + \frac{1}{p_m^{r}} \leq \prod_{i=m+1}^{\infty} \left( \sum_{j=0}^{1} \frac{1}{p_i^{r}} \right)$ for all positive integers $m$, then the range of $\sigma_r$ is dense in the interval $[1, \zeta(r))$.

Conversely, suppose that there exists some positive integer $m$ such that

$$1 + \frac{1}{p_m^{r}} > \prod_{i=m+1}^{\infty} \left( \sum_{j=0}^{1} \frac{1}{p_i^{r}} \right).$$

Fix some $N \in \mathbb{N}$, and let $N = \prod_{i=1}^{v} q_i^{\gamma_i}$ be the canonical prime factorization of $N$. If $p_s \not| N$ for some $s \in \{1, 2, \ldots, m\}$, then

$$\sigma_r(N) \geq 1 + \frac{1}{p_s^{r}} \geq 1 + \frac{1}{p_m^{r}}.$$

On the other hand, if $p_s \not| N$ for all $s \in \{1, 2, \ldots, m\}$, then

$$\sigma_r(N) = \prod_{i=1}^{v} \sigma_r(q_i^{\gamma_i}) = \prod_{j=1}^{v} \left( \sum_{j=0}^{\gamma_i} \frac{1}{q_i^{j}} \right).$$
Lemma 2.1. If we need a short lemma for the range of $p$ interval for all $m \geq 1 + 1$

Theorem 2.2. Proof. Pierre Dusart [1] has shown that, for $396738$

Let us assume that $F$ was arbitrary, this shows that there is no element of the range of $\sigma_{-r}$ is not dense in $[1, \zeta(r))]$.

Theorem 2.1 provides us with a method to determine values of $r > 1$ with the property that the range of $\sigma_{-r}$ is dense in $[1, \zeta(r))]$. However, doing so is still a somewhat difficult task. Luckily, for $r \in (1, 2]$, we may greatly simplify the problem with the help of the following theorem. First, we need a short lemma.

Lemma 2.1. If $j \in \mathbb{N}\{1, 2, 4\}$, then $\frac{p_{j+1}}{p_{j}} < \sqrt{2}$.

Proof. Pierre Dusart [1] has shown that, for $x \geq 396738$, there must be at least one prime in the interval $[x, x + \frac{x}{25 \log^2 x}]$. Therefore, whenever $p_{j} > 396738$, we may set $x = p_{j} + 1$ to get $p_{j+1} \leq (p_{j} + 1) + \frac{p_{j} + 1}{25 \log^2 (p_{j} + 1)} < \sqrt{2}p_{j}$. Using Mathematica 9.0 [3], we may quickly search through all the primes less than 396738 to conclude the desired result.

Theorem 2.2. Let $r$ be a real number in the interval $(1, 2]$. The range of $\sigma_{-r}$ is dense in the interval $[1, \zeta(r)]$ if and only if $1 + \frac{1}{P_{1}} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_{i}^{r}}\right)$ for all $m \in \{1, 2, 4\}$.

Proof. Let $F(m, r) = \left(1 + \frac{1}{P_{m}^{r}}\right) \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_{i}^{r}}\right)$ so that the inequality $1 + \frac{1}{P_{m}^{r}} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_{i}^{r}}\right)$ is equivalent to $F(m, r) \leq \zeta(r)$. In light of Theorem 2.1, it suffices to show that if $F(m, r) \leq \zeta(r)$ for all $m \in \{1, 2, 4\}$, then $F(m, r) \leq \zeta(r)$ for all $m \in \mathbb{N}$. Thus, let us assume that $r$ is such that $F(m, r) \leq \zeta(r)$ for all $m \in \{1, 2, 4\}$. If $m \in \mathbb{N}\{1, 2, 4\}$, then Lemma 2.1 tells us that $\frac{p_{m+1}}{p_{m}} < \sqrt{2} \leq \sqrt{2}$, which implies that $-\frac{2}{p_{m+1}} > \frac{1}{p_{m}}$. We then have

$$F(m + 1, r) = \left(1 + \frac{1}{P_{m+1}^{r}}\right) \prod_{i=1}^{m+1} \left(\sum_{j=0}^{\infty} \frac{1}{p_{i}^{r}}\right) > \left(1 + \frac{1}{P_{m+1}^{r}}\right)^{2} \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_{i}^{r}}\right)$$

$$> \left(1 + \frac{2}{p_{m+1}^{r}}\right) \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_{i}^{r}}\right) > \left(1 + \frac{1}{p_{m}^{r}}\right) \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_{i}^{r}}\right) = F(m, r)$$

for all $m \in \mathbb{N}\{1, 2, 4\}$. This means that $F(3, r) < F(4, r) \leq \zeta(r)$. Furthermore, $F(m, r) < \zeta(r)$ for all integers $m \geq 5$ because $(F(m, r))_{m=3}$ is a strictly increasing sequence and $\lim_{m \to \infty} F(m, r) = \zeta(r)$.
We have seen that, for \( r \in (1, 2] \), the range of \( \sigma_r \) is dense in \([1, \zeta(r)]\) if and only if \( F(m, r) \leq \zeta(r) \) for all \( m \in \{1, 2, 4\} \). Using Mathematica 9.0, one may plot a function \( g_m(r) = F(m, r) - \zeta(r) \) for each \( m \in \{1, 2, 4\} \). It is then easy to verify that \( g_2 \) has precisely one root, say \( \eta \), in the interval \((1, 2]\) (for anyone seeking a more rigorous proof of this fact, we mention that it is fairly simple to show that \( g_2'(r) > 0 \) for all \( r \in (1, 2] \)). Furthermore, one may confirm that \( g_1(r), g_2(r), g_4(r) \leq 0 \) for all \( r \in (1, \eta] \) and that \( g_2(r) > 0 \) for all \( r \in (\eta, 3] \). Hence, we have proven (or at least left the reader to verify) the first part of the following theorem.

**Theorem 2.3.** Let \( \eta \) be the unique number in the interval \((1, 2]\) that satisfies the equation

\[
\left( \frac{2^n}{2^n - 1} \right) \left( \frac{3^n + 1}{3^n - 1} \right) = \zeta(\eta).
\]

If \( r \in (1, \infty) \), then the range of the function \( \sigma_r \) is dense in the interval \([1, \zeta(r)]\) if and only if \( r \leq \eta \).

**Proof.** In virtue of the preceding paragraph, we know from the fact that

\[ g_2(\eta) = F(2, \eta) - \zeta(\eta) = \left( \frac{2^n}{2^n - 1} \right) \left( \frac{3^n + 1}{3^n - 1} \right) - \zeta(\eta) = 0 \]

that if \( r \in (1, 3] \), then the range of \( \sigma_r \) is dense in \([1, \zeta(r)]\) if and only if \( r \leq \eta \). We now show that the range of \( \sigma_r \) is not dense in \([1, \zeta(r)]\) if \( r > 3 \). To do so, we merely need to show that \( F(1, r) > \zeta(r) \) for all \( r > 3 \). For \( r > 3 \), we have

\[
F(1, r) = \left( 1 + \frac{1}{2^r} \right) \sum_{j=0}^{\infty} \frac{1}{2^{jr}} > \left( 1 + \frac{1}{2^r} \right)^2 = 1 + \frac{1}{2^r} + \frac{3}{4} \left( \frac{1}{2^{r-1}} \right) > 1 + \frac{1}{2^r} + \frac{1}{(r-1)2^{r-1}} = 1 + \frac{1}{2^r} + \int_{2}^{\infty} \frac{1}{x^r} dx > \zeta(r).
\]

\[\square\]

### 3 An open problem

We end by acknowledging that it might be of interest to consider the number of “gaps” in the range of \( \sigma_r \) for various \( r \). For example, for which values of \( r \in (1, \infty) \) is there precisely one gap in the range of \( \sigma_r \)? More generally, if we are given a positive integer \( L \), then, for what values of \( r > 1 \) is the closure of the range of \( \sigma_r \) a union of exactly \( L \) disjoint subintervals of \([1, \zeta(r)]\)?

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