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Trigonometric Pseudo Fibonacci Sequence

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Abstract: In this paper, we establish some results about second order non homogeneous recurrence relation containing extended trignometric function. Earlier [4], we proved some properties of recurrence relation

$$g_{n+2} = g_{n+1} + g_n + At^n, n = 0, 1, ...$$

with $g_0 = 0, g_1 = 1$, where both $A \neq 0$ and $t \neq 0$, and also $t \neq \alpha$, β, where α, β are the roots of $x^2 - x - 1 = 0$.

Using the properties of generalised circular functions and Elmore's method, we define a new sequence $\{H_n\}$ which is the extension of Pseudo Fibonacci Sequence, given by recurrence relation

$$H_{n+2} = pH_{n+1} - qH_n + Rt^n N_{r,0}(t^*x),$$

where $N_{r,0}(t^*x)$ is extended circular function.

We state and prove some properties for this extended Pseudo Fibonacci Sequence $\{H_n\}$.

Keywords: Pseudo Fibonacci Sequence, Non-homogeneous recurrence relation.
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1 Introduction

The second order linear recurrence sequence $\{w_n\} = \{w_n(a, b; p, q)\}$ was defined by A. F. Horadam [3], given by the relation

$$w_{n+2} = pw_{n+1} + qw_n, \ n \geq 0$$

with

$$w_0 = a \ and \ w_1 = b.$$

This sequence was used to generalise many sequences such as Fibonacci sequence, Lucas sequence, Pell sequence, Pell Lucas sequence, Jacobsthal sequence.

In this paper we generalise the Pseudo Fibonacci Sequence defined in [4]. We now extend this Sequence by taking the recurrence relation as

$$G_{n+2} = pG_{n+1} - qG_n + At^n,$$

where both $A \neq 0$ and $t \neq 0$, and also $t \neq \alpha$, β, where α, β are the roots of $x^2 - px + q = 0$.

We state and prove some identites for the Sequence $\{G_n\}$.

Further we extend this sequence to get more generalise sequence using circular functions.

Definition: We define Generalized Pseudo Fibonacci Sequence $\{G_n\}$ as the sequence satisfying the following non-homogeneous recurrence relation

$$G_{n+2} = pG_{n+1} - qG_n + At^n, n \geq 0, A \neq 0 \quad and \quad t \neq 0, \alpha, \beta. \tag{1.1}$$

with $G_0 = a$ and $G_1 = b$. Here a, b, p, q are arbitrary integers. First few initial terms of $\{G_n\}$ are given below:

$$G_2 = pb - qa + A$$
$$G_3 = p^2b - pqa - qb + A(p + t)$$
$$G_4 = p^3b - p^2qa - 2pqb + q^2a + A(p^2 + pt - q + t^2).$$

2 Some fundamental identities of $G_n(x)$

(i) Binets Formula: Let

$$B = B(t) = \frac{A}{t^2 - pt + q}. \tag{2.1}$$

Then the explicit Binet form of G_n is given by

$$G_n = c_1\alpha^n + c_2\beta^n + Bt^n, \tag{2.2}$$

where

$$c_1 = \frac{(b - a\beta) - B(t - \beta)}{\alpha - \beta}, \tag{2.3}$$

and

$$c_2 = \frac{(a\alpha - b) - B(\alpha - t)}{\alpha - \beta}, \tag{2.4}$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \qquad and \qquad \beta = \frac{p - \sqrt{p^2 - 4q}}{2} \tag{2.5}$$

are the roots of $x^2 - px + q = 0$. From (2.2) and (2.3), we deduce that,
$c_1 + c_2 = a - B$ and $c_1 - c_2 = \dfrac{(2b - ap) - B(2t - p)}{\alpha - \beta}$.

(ii) Generating function:
Generating function $G^*(x)$ for G_n is given as

$$G^*(x) = \sum_{n=0}^{\infty} G_n x^n.$$

$$= \frac{1}{(1 - px + qx^2)} \left(\frac{Ax^2}{1 - tx} + (a + bx - apx) \right). \tag{2.6}$$

(iii) Exponential Generating function:
Exponential Generating function $E^*(x)$ for G_n is given as

$$E^*(x) = \sum_{n=0}^{\infty} \frac{G_n x^n}{n!}.$$

$$= c_1 e^{\alpha x} + c_2 e^{\beta x} + Be^{tx}. \tag{2.7}$$

where c_1 and c_2 are the as in (2.3) and (2.4) respectively.
$E * (x)$ redues to exponential generating function for Fibonacci Sequence $\{W_n\}$ [6], if $B = 0, p = 1, q = -1; a = 0$ and $b = 1$.

(iv) $\lim\limits_{n\to\infty} \frac{G_n}{G_{n-1}} = \alpha$, if $|t/\alpha| < 1$.

(v) $\lim\limits_{n\to\infty} \frac{G_n}{G_{n-k}} = \alpha^k$, if $|t/\alpha| < 1$.

(vi) $\sum\limits_{k=0}^{n} G_k = \dfrac{1}{(p - q - 1)} \left[G_{n+1} - qG_n - b + (p - 1)a - A \sum\limits_{k=0}^{n-1} t^k \right]$.

(vii) $\sum\limits_{k=0}^{n-1} (-1)^k G_k$
$= \dfrac{1}{(p - q + 1)} \left[(-1)^{n+1} G_{n+1} + b - (p + 1)(a + (-1)^{n+1} G_n) + A(1 - t) \sum\limits_{k=0}^{n-1} (-1)^k t^{2k} \right]$.

(viii) $\sum\limits_{k=0}^{n} G_k t^k = \dfrac{1}{(1 - pt + qt^2)} \left[a(1 - pt) + bt - G_{n+1} t^{n+1} + qG_n t^{n+2} + A \sum\limits_{k=0}^{n} t^{2k} \right]$.

We prove identity (viii).

Proof: By recurrence relation, we have

$$G_n t^n = pG_{n-1} - qG_{n-2} + At^{n-2}t^n$$
$$G_{n-1}t^{n-1} = [pG_{n-2} - qG_{n-3} + At^{n-3}]t^{n-1}$$
$$\vdots$$
$$G_2 t^2 = [pG_1 - qG_0 + At^0]t^2$$

and on summing both sides we get

$$\sum_{k=2}^{n} G_k t^k = p\sum_{k=1}^{n-1} G_k t^{k+1} - q\sum_{k=0}^{n-2} G_k t^{k+2} + A\sum_{k=2}^{n} t^{2k-2}.$$

Therefore,

$$\sum_{k=0}^{n} G_k t^k = G_0 + G_1 t + pt\sum_{k=1}^{n-1} G_k t^k - qt^2\sum_{k=0}^{n-2} G_k t^k + A\sum_{k=2}^{n} t^{2k-2}.$$

Hence,

$$\sum_{k=0}^{n} G_k t^k(1-pt+qt^2) = G_0 + G_1 t + pt(-G_0 - G_n t^n) - qt^2(-G_{n-1}t^{n-1} - G_n t^n) + A\sum_{k=2}^{n} t^{2k-2}.$$

Therefore,

$$\sum_{k=0}^{n} G_k t^k = \frac{1}{(1-pt+qt^2)}\left[G_0(1-pt) + G_1 t - (G_{n+1} - At^{n-1})t^{n+1} + qG_n t^{n+2} + A\sum_{k=2}^{n} t^{2k-2}\right].$$

$$=\frac{1}{(1-pt+qt^2)}\left[G_0(1-pt) + G_1 t - G_{n+1}t^{n+1} + qG_n t^{n+2} + A\sum_{k=0}^{n} t^{2k}\right].$$

Hence,

$$\sum_{k=0}^{n} G_k t^k = \frac{1}{(1-pt+qt^2)}\left[a(1-pt) + bt - G_{n+1}t^{n+1} + qG_n t^{n+2} + A\sum_{k=0}^{n} t^{2k}\right].$$

This completes the proof. □

3 Extension using Elmore's Method

We generalise sequence $\{G_n\}$ by applying Elmore's Method [1] as follows.
Let

$$E_0(x) = E^*(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} + Be^{tx},$$

as in (2.7). On differentiating $E^*(x)$ n times with respect to x, we get a sequence $\{E_n(x)\}$ of $E_n(x)$ where

$$E_0^{(n)}(x) = E_n(x) = c_1\alpha^n e^{\alpha x} + c_2\beta^n e^{\beta x} + Bt^n e^{tx}. \tag{3.1}$$

It is interesting to note that $E_n(x)$ satisfy the non-homogeneous recurrence relation

$$E_{n+2} = pE_{n+1} - qE_n + Ae^{tx}t^n.$$

4 Extended Circular Functions

The Generalized Circular Functions defined by Mikusinsky [2] are as follows.

Let:

$$N_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, ..., r-1; \quad r \geq 1, \tag{4.1}$$

$$M_{r,j}(t) = \sum_{n=0}^{\infty} (-1)^r \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, ..., r-1; \quad r \geq 1. \tag{4.2}$$

Observe that
$N_{1,0}(t) = e^t, N_{2,0}(t) = cosht, N_{2,1}(t) = sinht$ and
$M_{1,0}(t)$= e^{-t}, $M_{2,0}(t)$= $cost$, $M_{2,1}(t)$= $sint$.

One obtains following result by differentiating (4.1) term by term with respect to t:

$$N_{r,j}^{(p)}(t) = \begin{cases} N_{r,j-p}(t), & 0 \leq p \leq j \\ N_{r,r+j-p}(t), & 0 \leq j < j < p \leq r \end{cases} \tag{4.3}$$

In particular, note from (4.3) that

$$N_{r,0}^{(r)}(t) = N_{r,0}(t),$$

so that in general

$$N_{r,0}^{(nr)}(t) = N_{r,0}(t), r \geq 1. \tag{4.4}$$

Further note that

$$N_{r,0}(0) = N_{r,0}^{(nr)}(0) = 1.$$

5 Generalisation of $\{G_n\}$ using Extented Circular Functions

Using Generalized Circular Functions and the technique of extension used in [5] we define the sequence $\{H_n(x)\}$ as follows. Let

$$H_0(x) = c_1 N_{r,0}(\alpha^* x) + c_2 N_{r,0}(\beta^* x) + R N_{r,0}(t^* x), \tag{5.1}$$

where α=$\alpha^{1/r}$, $\beta^* = \beta^{1/r}$ and $t^* = t^{1/r}$, r being the positive integer.
α, β are the roots of $x^2 - px + q = 0$. we have

$$\alpha + \beta = p, \alpha\beta = q. \tag{5.2}$$

Now, we define the sequence $\{H_n(x)\}$ successively as follows: $H_1(x) = H_0^{(r)}(x)$, $H_2(x) = H_0^{(2r)}(x)$, and in general $H_n(x) = H_0^{(nr)}(x)$, where derivatives are with respect to x.

Then from (5.1) and using (4.4) we get

$$H_1(x) = c_1 \alpha N_{r,0}(\alpha^* x) + c_2 \beta N_{r,0}(\beta^* x) + B t N_{r,0}(t^* x),$$

$$H_2(x) = c_1\alpha^2 N_{r,0}(\alpha^* x) + c_2\beta^2 N_{r,0}(\beta^* x) + Bt^2 N_{r,0}(t^* x),$$

and in general

$$H_n(x) = c_1\alpha^n N_{r,0}(\alpha^* x) + c_2\beta^n N_{r,0}(\beta^* x) + Bt^n N_{r,0}(t^* x). \tag{5.3}$$

Theorem 1. *The sequence $\{H_n(x)\}$ satisfies the non-homogeneous recurrence relation*

$$H_{n+2}(x) = pH_{n+1}(x) - qH_n(x) + At^n N_{r,0}(t^* x). \tag{5.4}$$

Proof. R.H.S. $= p\{c_1\alpha^{n+1} N_{r,0}(\alpha^* x) + c_2\beta^{n+1} N_{r,0}(\beta^* x) + Bt^{n+1} N_{r,0}(t^* x)\}$

$$q\{c_1\alpha^n N_{r,0}(\alpha^* x) + c_2\beta^n N_{r,0}(\beta^* x) + Bt^n N_{r,0}(t^* x)\} + At^n N_{r,0}(t^* x)$$
$$= c_1\alpha^n N_{r,0}(\alpha^* x)\{p\alpha - q\} + c_2\beta^n N_{r,0}(\beta^* x)\{p\beta - q\} + t^n N_{r,0}(t^* x)\{Bt - B + A\}. \tag{5.5}$$

Using the fact that α and β are the roots of $x^2 - px + q = 0$ and (2.1) in (5.2) we get,

$$R.H.S. = c_1\alpha^{n+2} N_{r,0}(\alpha^* x) + c_2\beta^{n+2} N_{r,0}(\beta^* x) + pt^{n+2} N_{r,o}(t^* x).$$
$$= H_{n+2}(x).$$

□

6 Reduction to Fibonacci Sequence

Observe that, for $r = 1$, $\alpha^* = \alpha$, $\beta^* = \beta$ and $N_{r,0}(t) = N_{1,0}(t) = e^t$, (5.3) becomes

$$\begin{aligned} H_n(x) &= c_1\alpha^n e^{\alpha x} + c_2\beta^n e^{\beta x} + Bt^n e^{tx} \\ &= E_n(x). \end{aligned} \tag{6.1}$$

In addition to above particular value of $r, p = 1, q = -1, a = 0, b = 1, B = 0$ and $x = 0$, we use (6.1), to obtain

$$H_n(0) = E_n(0) = F_n.$$

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