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Pell and Lucas primes

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Abstract: The structures of Pell and Lucas numbers, P_p and L_p with prime subscripts are compared in relation to the function ($Kp \pm 1$) and for factors of the form ($kp \pm 1$). It is found that digit sums give some guides to primality.
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1 Introduction

We have recently considered some methods for checking the primality of Fibonacci numbers [5–9]. For some of the earlier history see [1]. Since the Pell and Lucas numbers are structurally similar to the Fibonacci numbers it makes sense to attempt to apply the same methods to these numbers. The Pell numbers are generated from the second order linear homogeneous recurrence relation

$$P_n = 2P_{n-1} + P_{n-2},\ n \geq 0. \tag{1.1}$$

and with suitable initial terms we obtain (Table 1).

n	***P***$_n$	***n***	***P***$_n$	***n***	***P***$_n$	***n***	***P***$_n$
0	0	11	5741	21	38613965	31	259717522849
1	1	12	13860	22	93222358	32	627013566048
2	2	13	33461	23	225058681	33	1513744654945
3	5	14	80782	24	543339720	34	3654502875938
4	12	15	195025	25	1311738121	35	88227550406821
5	29	16	470832	26	3166815962	36	21300003689580
6	70	17	1136689	27	7645370045	37	51422757785981
7	169	18	2744210	28	18457556052	38	124145519261542
8	408	19	6625109	29	44560482149	39	299713796309065
9	985	20	15994428	30	107578520350	40	723573111879672
10	2378					41	1746860020068409

Table 1: Some Pell numbers

2 Factor functions

Unlike the Lucas numbers, the Pell numbers [16] satisfy

$$F_n = \left(F_{\frac{n+1}{2}} \right)^2 + \left(F_{\frac{n+1}{2}} \right)^2; \tag{2.1}$$

that is,

$$P_n = \left(\tfrac{1}{2} P_{n+1}\right)^2 + \left(\tfrac{1}{2} P_{n-1}\right)^2. \tag{2.2}$$

It might be expected then that the Fibonacci factor function

$$f\left(F_p\right) = kp \pm 1 \tag{2.3}$$

would also extend to composite prime-subscripted Pell numbers – which it does (Table 2).

p	P_p	Digit sum	Type	Factors
3	5	5	p	---
5	29	2	p	---
7	169	7	c	13 × 13; 13 = 2p – 1
11	5741	8	p	---
13	33461	8	p	---
17	1136689	7	c	137 × 8297; 137 = 8p + 1; 8297 = 488p + 1
19	6625109	2	p	---
23	225058681	1	c	229 × 982789; 229 = 10p + 1; 982789 = 42730p – 1
29	44560482149	2	p	---
31	259717522849	7	c	61 × 4257664309; 61 = 2p – 1; 4257664309 = 137344010p – 1
37	51422757785981	8	p	---
41	1746860020068409	7	p	---

Table 2: Factors of some prime-subscripted Pell numbers
[Factorisations checked with Mathematica and WolframAlpha]

The prime-subscripted Pell numbers in Table 2 also satisfy the recurrence relation

$$P_{2n+1} = 6P_{2n-1} - P_{2n-3},\ n > 1. \tag{2.4}$$

with initial conditions 1 and 5. They are also worthy of note in this context because they are related to the Pell identities of Horadam, [3] and the Pythagorean triads of Forget and Larkin, [2] through related Pell-type sequences $\{R_n\}$ and $\{S_n\}$. For example,

$$2P_{2n-1}^2 = R_n^2, \tag{2.5}$$

in which $\{R_n\}$ also satisfies (2.5) but with initial conditions $R_1 = 1$ and $R_2 = 7$. From (2.5) we can also obtain

$$P_{2n+5} = P_{2n+1} + 4R_n, \tag{2.6}$$

and

$$P_{2n+4} = P_{2n} + 4S_{n+2}, \tag{2.7}$$

where $\{S_n\}$ also satisfies (2.7) but with initial conditions $S_1 = 1$ and $S_2 = 3$, so that

$$2S_n = R_n - R_{n-1}. \tag{2.8}$$

It is also worthy of note that numbers which satisfy (2.4) with initial conditions (1, 6) or (3, 17) are called balancing numbers: an integer n is called a balancing number (or a Lucas-balancing number) with balancer r if it is the solution of the Diophantine equation [12]

$$\sum_{j=1}^{n-1} j = \sum_{j=n+1}^{n+r} j.$$

For example,

$$1 + 2 + 3 + 4 + 5 = 15 = 7 + 8,$$

so that 6 is a balancing number with balancer 2.

3 *K* function

The Fibonacci number, F_p, may be expressed as

$$F_p = Kp \pm 1 \tag{3.1}$$

in which K is a function of the sum of p consecutive Fibonacci numbers [9]. The digit sum of K yields a primality check [8]. Correspondingly (Table 3)

$$P_p = Kp \pm 1 \tag{3.2}$$

p	$p \vee c$	K	$Kp + 1$	$Kp - 1$
3	p	2		✓
5	p	6		✓
7	c	24	✓	
11	p	522		✓
13	p	2574		✓
17	c	66864	✓	
19	p	348690		✓
23	c	9785160	✓	
29	p	1536568350		✓
31	c	8377984608	✓	
37	p	1389804264486		✓
41	p	42606341952888	✓	

Table 3: Unit signs in Equation (3.2)

The sign for 1 in Equation (3.2) is generally negative for primes and positive for composites for the range considered here. The values for the digit sum of K for Pell numbers exhibit a common sum when the right-end-digit (RED) (or value (modulo 10)) $p^* = 3$, so K is not as consistent as it is for F_p (Table 4).

p^*	p	c
1	3,9	6
3	2,9	9
7	9	3,6
9	3,6	---

Table 4: Sum of digit of K (from Equation (3.2))

p^*	p	c
1	7,8	7
3	5,8	1
7	8	7
9	2	---

Table 5: Sum of digits of P_p (from Table 1)

Furthermore, the digit sums of P_p show clearer distinctions than K between primes and composites (Table 5).

4 Lucas primes

The Lucas numbers are similar in structure to the Fibonacci numbers in many ways, but particularly in the context of this paper in that they consist of the odd-odd-even form, whereas the Pell numbers follow an even-odd pattern. However, the Lucas primes are not restricted to Lucas numbers with a prime subscript; for example, $L_8 = 47$ and $L_{16} = 2207$, both of which are prime numbers. On the other hand, the number

$$L_n = Kn \pm 1 \tag{4.1}$$

is essentially the same as for F_p and P_p. For instance,

$$L_8 = 6n - 1 = 47; \; L_{16} = 138n - 1 = 2207; \; L_{11} = 18n + 1 = 199.$$

Moreover, for composite L_p the factors have the pattern ($kp \pm 1$) as for the Fibonacci and Pell sequences, [8]. For example,

$$L_{23} = 139 \times 461 = (6p+1)(20p+1).$$

Why then do the elements of the Lucas sequence fail to follow Equations (2.2) or (2.3)? None of the Lucas numbers have REDs equal to 0 or 5; that is, none is in the class $\overline{0}_5 \subset Z_5$ [10], which means that constraints occur.

- If $L_n^* \in \{3, 7\}$, then the values of ½(n + 1) and $n(n - 1)$ cannot be integers as n is always even, so that L_n with these REDs cannot satisfy Equations (2.2) or (2.3);
- If $L_n^* \in \{1, 9\}$, then the values of ½(n + 1) and ½(n − 1) yield d^2, e^2 couples of (1, 4) or (9, 6) and L_n REDs are 5 so that Equations (2.2) and (2.3) cannot be satisfied.

Primes have one set of (d,e) with no common factors [4] and for Pell and Fibonacci numbers this set is given by Equations (2.2) and (2.3). Composites have as many (d, e) couples as there are factors. These restrictions permit a further test for primality, which is discussed in detail elsewhere [4], and this structure explains why some primes, $L_n^* = 3, 7$ have even *n*.

5 Final comments

It has been found here that the Lucas numbers show differences from the Pell and Fibonacci numbers in the formation of primes. The results are related to those of Mc Daniel [11], which build on the results of Robbins [14, 15] and are related to Ribenboim's, [13]. The results here provide further insights into the structure of the prime system. The functions $(pK \pm 1)$ and $(pk \pm 1)$ are independent of the triples formed from the Fibonacci, Pell and Lucas sequences.

The overall prime structure which applies to the subscripts of the three sets of triples must relate to the pattern of primes versus composites formed. In other words, here are clues to the master prime structure. Further analysis along the lines we have been following [4–9] might well link up with other prime-structure research.

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