Tridiagonal matrices related to subsequences of balancing and Lucas-balancing numbers

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Abstract: It is well known that balancing and Lucas-balancing numbers are expressed as determinants of suitable tridiagonal matrices. The aim of this paper is to express certain subsequences of balancing and Lucas-balancing numbers in terms of determinants of tridiagonal matrices. Using these tridiagonal matrices, a factorization of the balancing numbers is also derived.

Keywords: Balancing numbers, Balancers, Lucas-balancing numbers, Tridiagonal matrices.

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1 Introduction

The concept of balancing numbers was originally introduced by Behera et.al [1] in connection with the Diophantine equation

\[ 1 + 2 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + r), \]

where, they call \( n \) a balancing number and \( r \) the balancer corresponds to \( n \). The sequence of balancing number \( \{B_n\} \) satisfy the recurrence relations

\[ B_{n+1} = 6B_n - B_{n-1}, \ n \geq 2, \]

with \( B_1 = 1, B_2 = 6 \) and \( B_{n+1} = \frac{B_n^2 - 1}{B_{n-1}}, \ n \geq 2. \) The number sequence closely associates to the balancing numbers is the sequence of Lucas-balancing numbers whose recurrence relation is

\[ C_{n+1} = 6C_n - C_{n-1}, \ n \geq 2, \]
with $C_1 = 3$, $C_2 = 17$, where $C_n$ denotes the $n^{th}$ Lucas-balancing number. The closed form (popularly known as Binet’s formula) for both balancing and Lucas-balancing numbers are respectively given by $B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$ and $C_n = \frac{\lambda_1^n + \lambda_2^n}{2}$ where $\lambda_1 = 3 + \sqrt{8}$, $\lambda_2 = 3 - \sqrt{8}$ [1, 7].

Panda [8] has shown that, the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers. In [7], Panda and Ray have proved that the Lucas-balancing numbers are nothing but the even ordered terms of the associated Pell sequence. Also they have shown that the $n^{th}$ balancing number is product of $n^{th}$ Pell number and $n^{th}$ associated Pell numbers. Liptai, et al. [5] added another interesting result to the theory of balancing numbers by generalizing these numbers. A. Berczes, et al. [3] and P. Olajos [6] surveyed some interesting properties and results on generalized balancing numbers. Recently, many interesting results on balancing numbers and their related sequences are studied by different authors [2, 4, 10, 12, 13, 14, 15, 16, 17]. There is another way to generate balancing numbers through matrices called as balancing matrices which were introduced in [11].

There are many connections between balancing, Lucas-balancing numbers and tridiagonal matrices. In [9], Ray introduced a family of tridiagonal matrices of order $n$

$$D(n) = \begin{pmatrix} 6 & -i & & & \\ i & 6 & -i & & \\ & i & 6 & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & 6 \end{pmatrix},$$

whose determinants $|D(k - 1)|$ are nothing but the balancing numbers $B_k$, starting with $k = 2$. Replacing the first row first column entry of $D(n)$ by 3 to get

$$M(n) = \begin{pmatrix} 3 & -i & & & \\ i & 6 & -i & & \\ & i & 6 & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & 6 \end{pmatrix},$$

whose determinants now generate the Lucas-balancing numbers $C_k$, starting with $k = 1$. In this paper, we extend these results to construct families of tridiagonal matrices whose determinants generate any arbitrary linear sub-sequences $B_{\alpha k + \beta}$ or $C_{\alpha k + \beta}$ of the balancing and Lucas-balancing numbers with $k \in \mathbb{Z}^+$. We choose a specific linear subsequence of balancing numbers and use it to derive the factorization

$$B_{2mn} = B_{2m} \prod_{1 \leq k \leq n-1} 2 \left( C_{2m} - \cos \frac{k\pi}{n} \right).$$

The factorization (3) indeed, a generalization of one of

$$B_n = \prod_{1 \leq k \leq n-1} 2 \left( 3 - \cos \frac{k\pi}{n} \right),$$

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given in [9]. In order to derive (3), we present the following known result which describes the sequence of determinants for a general tridiagonal matrix (see Theorem 1.1, [9]).

**Theorem 1.1.** If the family of tridiagonal matrices \( A(n) \), \( n = 1, 2, \ldots \) is of the form

\[
A(n) = \begin{pmatrix}
  a_{11} & a_{12} & & & \\
  a_{21} & a_{22} & a_{23} & & \\
  & a_{32} & a_{33} & \ddots & \\
  & & \ddots & \ddots & a_{(n-1)n} \\
  & & & a_{n(n-1)} & a_{nn}
\end{pmatrix},
\]

then, the successive determinants of \( A_n \) are given by the recursive formulas:

\[
|A(1)| = a_{11} \\
|A(2)| = a_{11}a_{22} - a_{12}a_{21} \\
|A(n)| = a_{nn}|A(n-1)| - a_{(n-1)n}a_{n(n-1)}|A(n-2)|.
\]

## 2 Balancing subsequences

By virtue of Theorem 1.1, we can generalize the families of tridiagonal matrices given by (1) and (2). For every linear subsequence of balancing numbers, we construct a family of tridiagonal matrices whose successive determinants are shown in the following result.

**Theorem 2.1.** The symmetric tridiagonal family of matrices \( D_{\alpha,\beta}(k) \), \( k = 1, 2, \ldots \), whose entries are

\[
d_{1,1} = B_{\alpha+\beta}, \ d_{2,2} = \left\lceil \frac{B_{2\alpha+\beta}}{B_{\alpha+\beta}} \right\rceil; \ d_{j,j} = 2C_{\alpha}, \ 3 \leq j \leq k, \\
d_{1,2} = d_{2,1} = \sqrt{d_{2,2}\cdot B_{\alpha+\beta} - B_{2\alpha+\beta}}; \ d_{j,j+1} = d_{j+1,j} = 1, \ 2 \leq j \leq k
\]

with positive integers \( \alpha \) and natural numbers \( \beta \), has successive determinants \( |D_{\alpha,\beta}(k)| = B_{\alpha k+\beta} \).

In order to prove Theorem 2.1, we need the following lemma.

**Lemma 2.2.** For all positive integers \( k \) and \( n \), \( B_{k+n} = 2C_n B_k - B_{k-n} \).

**Proof.** Using Binet’s formulas for \( B_n, C_n \) and since \( \lambda_1\lambda_2 = 1 \), we obtain

\[
2C_n B_k - B_{k-n} = (\lambda_1^n + \lambda_2^n) \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} - \frac{\lambda_1^{k-n} - \lambda_2^{k-n}}{\lambda_1 - \lambda_2}
\]

\[
= \lambda_1^{n+k} \lambda_2^{k-n} + \frac{\lambda_1^k \lambda_2^n - \lambda_1^{n+k} \lambda_2^{k-n} - \lambda_1^{k-n} \lambda_2^{n+k} + \lambda_2^{k-n}}{\lambda_1 - \lambda_2}
\]

\[
= \frac{\lambda_1^{n+k} - \lambda_2^{n+k}}{\lambda_1 - \lambda_2} = B_{k+n},
\]

which completes the proof. \( \square \)
Now we are in a position to prove Theorem 2.1.

**Proof.** (Theorem 2.1) We use method of induction to prove this result. Clearly, result holds for \( k = 1, 2 \).

\[
|D_{\alpha,\beta}(1)| = B_{\alpha+\beta}
\]

\[
|D_{\alpha,\beta}(2)| = \frac{B_{\alpha+\beta}}{\sqrt{d_{2,2}B_{\alpha+\beta} - B_{2\alpha+\beta}}} = B_{2\alpha+\beta}.
\]

Assume that, \(|D_{\alpha,\beta}(k)| = B_{\alpha k+\beta}\) for any natural number less than or equal to \( k \). Then by virtue of Theorem 1.1 and Lemma 2.2, we get

\[
|D_{\alpha,\beta}(k + 1)| = d_{k,k} |D_{\alpha,\beta}(k)| - d_{k,k-1} d_{k-1,k} |D_{\alpha,\beta}(k - 1)|
\]

\[
= 2C_{\alpha} |D_{\alpha,\beta}(k)| - |D_{\alpha,\beta}(k - 1)|
\]

\[
= 2C_{\alpha} B_{\alpha k+\beta} - B_{\alpha(k-1)+\beta} = B_{\alpha(k+1)+\beta},
\]

which ends the proof. \( \square \)

By Theorem 2.1, one can construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of balancing numbers. For example, the determinants of the following tridiagonal matrices

\[
\begin{pmatrix}
6 & 0 & 1 \\
0 & 1155 & 1 \\
1 & 1154 & \ddots \\
\vdots & \ddots & \ddots & 1 \\
1 & 1154 & \cdots & 1
\end{pmatrix},
\begin{pmatrix}
6930 & \sqrt{35} \\
\sqrt{35} & 198 & 1 \\
1 & 198 & \ddots \\
\vdots & \ddots & \ddots & 1 \\
1 & 198 & \cdots & 1
\end{pmatrix},
\begin{pmatrix}
40391 & \sqrt{1189} \\
\sqrt{1189} & 34 & 1 \\
1 & 34 & \ddots \\
\vdots & \ddots & \ddots & 1 \\
1 & 34
\end{pmatrix}
\]

are respectively given by \( B_{4k-2}, B_{3k+3} \) and \( B_{2k+5} \).

### 3 Lucas-balancing subsequences

In the last section, we discussed the link between linear balancing subsequences and tridiagonal matrices. In a similar manner, we can generalize the family of tridiagonal matrix (2) for linear subsequences of Lucas-balancing numbers. The following theorem demonstrates the claim.
**Theorem 3.1.** For every natural number $k$, the symmetric tridiagonal family of matrices $M_{\alpha,\beta}(k)$, whose entries are

\[
m_{1,1} = C_{\alpha+\beta}, \quad m_{2,2} = \frac{C_{2\alpha+\beta}}{C_{\alpha+\beta}}; \quad m_{j,j} = 2C_{\alpha}, \quad 3 \leq j \leq k;
\]

\[
m_{1,2} = m_{2,1} = \sqrt{m_{2,2}C_{\alpha+\beta} - C_{2\alpha+\beta}}; \quad m_{j,j+1} = m_{j+1,j} = 1, \quad 2 \leq j \leq k
\]

with positive integers $\alpha$ and natural numbers $\beta$, has successive determinants $|M_{\alpha,\beta}(k)| = C_{\alpha k+\beta}$.

Following result is required to prove the theorem. The proof of the lemma is omitted as it is analogous to Lemma 2.2.

**Lemma 3.2.** For any integers $k$ and $n$,

\[
C_{k+n} = 2C_{n}C_{k} - C_{k-n}.
\]

**Proof.** (Theorem 3.1) Once again, method of induction comes into the picture. Clearly, the result is true for $k = 1, 2$, since

\[
|M_{\alpha,\beta}(1)| = C_{\alpha+\beta}
\]

\[
|M_{\alpha,\beta}(2)| = \frac{C_{\alpha+\beta}}{\sqrt{m_{2,2}C_{\alpha+\beta} - C_{2\alpha+\beta}}}
\]

\[
\sqrt{m_{2,2}C_{\alpha+\beta} - C_{2\alpha+\beta}} = C_{2\alpha+\beta}.
\]

Assume that, $|M_{\alpha,\beta}(k)| = C_{\alpha k+\beta}$ for $1 \leq k \leq N$. Then, by Theorem 1.1 and Lemma 3.2, we have

\[
|M_{\alpha,\beta}(k+1)| = m_{k,k}|M_{\alpha,\beta}(k)| - m_{k,k-1}m_{k-1,k}|M_{\alpha,\beta}(k-1)|
\]

\[
= 2C_{\alpha}|M_{\alpha,\beta}(k)| - |M_{\alpha,\beta}(k-1)|
\]

\[
= 2C_{\alpha}C_{\alpha k+\beta} - C_{\alpha(k-1)+\beta} = C_{\alpha(k+1)+\beta}.
\]

which end the proof.

Using Theorem 3.1, one can construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of Lucas-balancing numbers. For example, the determinants of the following tridiagonal matrices

\[
\begin{pmatrix}
17 & 0 & \cdots & 1 \\
0 & 1153 & \cdots & 1 \\
1 & 1154 & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots \\
1 & 1154 & \cdots & 1 \\
\end{pmatrix},
\begin{pmatrix}
19601 & \sqrt{99} & \cdots & 1 \\
\sqrt{99} & 198 & \cdots & \cdots \\
1 & 198 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
1 & 198 & \cdots & 1 \\
\end{pmatrix},
\begin{pmatrix}
114243 & \sqrt{3363} & \cdots & 1 \\
\sqrt{3363} & 34 & \cdots & \cdots \\
1 & 34 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
1 & 34 & \cdots & 1 \\
\end{pmatrix}
\]

are respectively given by $C_{4k-2}, C_{3k+3}$ and $C_{2k+5}$. 

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4 Factorization of balancing numbers using tridiagonal matrices

In this section, we establish the factorization (3) presented in section 1. We consider the following symmetric tridiagonal matrices $T_m(n), \ \ n = 1, 2, \ldots$ where

$$T_m(n) = \begin{pmatrix} 2C_{2m}B_{2m} & \sqrt{B_{2m}} & 1 \\ \sqrt{B_{2m}} & 2C_{2m} & \ddots \\ & \ddots & \ddots & 1 \\ 1 & 2C_{2m} & \ddots & \ddots \\ & & \ddots & \ddots & 1 \end{pmatrix}. $$

In view of Lemma 2.2 and Lemma 3.2, we have, $2C_{2m}B_{2m} = B_{4m}$ and

$$\frac{B_{6m}}{B_{4m}} = \left[ \frac{2B_{4m}C_{2m} - B_{2m}}{B_{4m}} \right] = [2C_{2m} - B_{2m}] = 2C_{2m}. $$

Also,

$$\sqrt{\frac{B_{6m}}{B_{4m}}}B_{4m} - B_{6m} = \sqrt{2C_{2m}B_{4m} - B_{4m}} = \sqrt{B_{2m}}. $$

Thus, $T_m(n)$ is a specific example of the family of tridiagonal matrices $D_{\alpha,\beta}(n)$ with $\alpha = 2m$ and $\beta = 2m$. By Theorem 2.1, we get $|T_m(n)| = B_{2m(n+1)}$. If $e_j$ be the $j^{th}$ column of an $n \times n$ identity matrix, we have

$$|T_m(n)| = B_{2m}|R_m(n)|,$$

where $R_m(n) = \left( I + \left( \frac{1}{B_{2m}} - 1 \right)e_1e_1^T \right) T_m(n)$. If $\lambda_k, k = 1, 2, \ldots$ be the eigenvalues of $R_m(n)$ with corresponding eigenvectors $X_k$ and since the product of all eigenvalues is nothing but the value of the determinant, we get $|R_m(n)| = \prod_{k=1}^n \lambda_k$. We now introduce a new tridiagonal matrix $S_m(n) = R_m(n) - 2C_{2m}I$ and observe that,

$$S_m(n)X_k = R_m(n)X_k - 2C_{2m}X_k = \lambda_k X_k - 2C_{2m}X_k = (\lambda_k - 2C_{2m})X_k. $$

Thus, $\gamma_k = \lambda_k - 2C_{2m}$ are the eigenvalues of $S_m(n)$. An eigenvalue $\gamma$ of $S_m(n)$ is a root of the characteristic polynomial $|S_m(n) - \gamma I|$ and this polynomial can be transformed into Chebyshev polynomial of second kind [9], with roots $\gamma_k = -2\cos \frac{\pi k}{n+1}$. Therefore,

$$B_{2m(n+1)} = |T_m(n)| = B_{2m}|R_m(n)| = B_{2m} \prod_{k=1}^n \lambda_k = B_{2m} \prod_{k=1}^n \left( 2C_{2m} - 2\cos \frac{\pi k}{n+1} \right),$$

which follows the factorization (3) by simple change of variables.
References


