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A Möbius arithmetic incidence function

Emil Daniel Schwab[1], Gabriela Schwab[2]

[1] Department of Mathematical Sciences
The University of Texas at El Paso
El Paso, Texas 79968, USA
e-mail: eschwab@utep.edu

[2] Department of Mathematics
El Paso Community College
El Paso, Texas 79902, USA
e-mail: gschwab@epcc.edu

Abstract: The aim of this note is to study a non-standard right cancellative and half-factorial Möbius monoid, and to compute its Möbius function.
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1 Introduction

The classical settings for Möbius inversion are special cases of Leroux's Möbius categories. The locally finite posets are categories in which there is one morphism $x \to y$ whenever $x \leq y$; and the monoids with the finite decomposition property are categories with only one object. In [7] an arithmetic incidence function is a complex-valued function $\xi : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ defined for all pairs of positive integers such that $\xi(m, n) = 0$ if $m \not\leq n$. An arithmetic incidence function defined above has all the defining properties of both an arithmetic function of two variables and a poset incidence function. In this short note we consider a simple example where "poset incidence function" is replaced by "monoid incidence function".

The S-convolution ("S" from the standard ordering) considered in [7] as a generalization of the Cauchy convolution is defined by (see [7, Definition 5.2]):

$$(\forall (m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+): \quad (\xi * \eta)(m,n) = \sum_{\substack{m \le p \le n \\ p+q=m+n}} \xi(m,p)\eta(m,q).$$

Now, referring to the S-convolutions, we can view the arithmetic indidence functions as complex-valued functions defined on the set

$$M_{\le} = \{(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \ m \le n\}.$$

The set $M_{\le}$ contains the diagonal Δ of $\mathbb{Z}_+ \times \mathbb{Z}_+$ and a such arithmetic incidence function ξ has a convolution inverse if and only if $\xi(m,m) \neq 0$ for every $(m,m) \in \Delta$. The Möbius function μ is the convolution inverse of the zeta function ζ defined by $\zeta(m,n) = 1$ for all arguments of the domain set $M_{\le}$.

It is clear that if we remove the diagonal, for example instead of $M_{\le}$ we consider

$$M_{\lneqq} = \{(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \ m \lneqq n\},$$

then the tide turns and the problem of Möbius function (Möbius inversion) takes on another meaning. The set $M_{\lneqq}$ equipped with a convenient multiplication (denoted by $\cdot$ but often omitted) leads us to a monoid convolution of arithmetic functions of two variables. For example, if $M_{\lneqq}$ is a right cancellative monoid such that no element is invertible except the identity e, and for any $(u,v) \in M_{\lneqq}$ there are at most a finite number of pairs $((m,n),(p,q)) \in M_{\lneqq} \times M_{\lneqq}$ such that $(u,v) = (m,n) \cdot (p,q)$ (in other words, if $M_{\lneqq}$ is a right cancellative Möbius monoid), then one may define the convolution $\xi * \eta$ of two arithmetic functions $\xi, \eta : M_{\lneqq} \to \mathbb{C}$ by

$$(\xi * \eta)(u,v) = \sum_{(m,n)\cdot(p,q)=(u,v)} \xi(m,n)\eta(p,q).$$

In this case an arithmetic function of two variables ξ has a convolution inverse if and only if $\xi(e) \neq 0$ (see [5, Proposition 2.2]). Thus the diagonal of a locally finite partial ordered set was substituted by the identity element of the monoid; the Möbius function $\mu : M_{\lneqq} \to \mathbb{C}$ being the convolution inverse of the zeta function $\zeta : M_{\lneqq} \to \mathbb{C}$ ($\forall (u,v) \in M_{\lneqq},\ \zeta(u,v) = 1$).

The monoid $(M_{\lneqq}, \cdot)$ that we consider in this note is a special monoid: it is right cancellative but not left cancellative; it is atomic (with only two atoms) and all factorizations of a non-identity into atoms have the same length. It is also a Möbius monoid (a Möbius category in the sense of Leroux [1],[3] with one object; for more details see [2] or [5]), etc. As a non-standard example, it brings new challenges to the study of convolutions of such arithmetic incidence functions. The computation of the Möbius function is presented in the last Section.

2 The right cancellative Möbius monoid $M_{\lneqq}$

The set

$$M_{\lneqq} = \{(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ | m \lneqq n\}$$

equipped with the multiplication defined by:

$$(m,n)\cdot(p,q)=\begin{cases}(p,n+q-2) & \text{if} \quad m=1\\ (m+q-2,n+q-2) & \text{if} \quad m\geq 2\end{cases}$$

is a non-commutative monoid, the element $e=(1,2)$ being the identity. It is straightforward to check that this monoid is right cancellative but it is not left cancellative. We say that $y\in M_{\lneqq}$ is a right divisor of $z\in M_{\lneqq}$ if there exists a (necessary unique) $x\in M_{\lneqq}$ such that $z=x\cdot y$. We write in symbols $y|z$ and $x=\frac{z}{y}$. Since $M_{\lneqq}$ is a right cancellative monoid, the right divisibility on $M_{\lneqq}$ is a partial order relation on $M_{\lneqq}$. As we will see below (Proposition 2.1), for any $z\in M_{\lneqq}$ the set of all right divisors of z in $M_{\lneqq}$ is finite; the (non-commutative) convolution $\xi*\eta$ of two arithmetic functions of two variables ξ and η on $M_{\lneqq}$ (i.e. complex valued functions with domain $M_{\lneqq}$) being given by:

$$(\xi*\eta)(z)=\sum_{y|z}\xi(\frac{z}{y})\eta(y).$$

Proposition 2.1. *Let $z=(u,v)\in M_{\lneqq}$. The set $R(z)$ of all right divisors of z is the following one:*

$$R(z)=\begin{cases}\{(1,v),(1,v-1),...,(1,2)\} & \textit{if} \quad u=1,\\ \{(u,v),(u,v-1),...,(u,u+1)\}\cup\{(1,2)\}\cup\{(1,3),(2,3)\}\cup... & \\ ...\cup\{(1,u),(2,u),...,(u-1,u)\} & \textit{if} \quad u>1.\end{cases}$$

Proof. Let $(m,n)\cdot(p,q)=(u,v)$.

$a)$ If $u=1$ then $m=p=1$ and $(1,n+q-2)=(1,v)$, that is $n+q=v+2$. Since $n,q>1$ it follows:

$$\begin{aligned}&n=2,q=v \quad\Rightarrow\quad (1,2)\cdot(1,v)=(1,v)\\ &n=3,q=v-1 \Rightarrow (1,3)\cdot(1,v-1)=(1,v)\\ &\dots\\ &n=v,q=2 \quad\Rightarrow\quad (1,v)\cdot(1,2)=(1,v)\end{aligned}$$

$b)$ If $u>1$ and

$b_1)$ $m=1$, then $p=u$ and $n+q=v+2$. Since $n>1$ and $q>u$ it follows:

$$\begin{aligned}&n=2,q=v \qquad\Rightarrow\qquad (1,2)\cdot(u,v)=(u,v)\\ &n=3,q=v-1 \qquad\Rightarrow\qquad (1,3)\cdot(u,v-1)=(u,v)\\ &\dots\\ &n=v-u+1,q=u+1\Rightarrow(1,v-u+1)\cdot(u,u+1)=(u,v)\end{aligned}$$

$b_2)$ $m>1$, then $m+q=v+2$ and $n+q=v+2$. It follows:

$q=2\Rightarrow m=u,n=v$ and $p<q$ implies

$$p=1 \text{ and } (u,v)\cdot(1,2)=(u,v)$$

$q = 3 \Rightarrow m = u-1, n = v-1$ and $p < q$ implies

$$\begin{cases} p = 1 \text{ and } (u-1, v-1)\cdot(1,3) = (u,v) \\ p = 2 \text{ and} (u-1, v-1)\cdot(2,3) = (u,v) \end{cases}$$

...

$q = u \Rightarrow m = 2, n = v-u+2$ and $p < q$ implies

$$\begin{cases} p = 1 \text{ and } (2, v-u+2)\cdot(1,u) = (u,v) \\ p = 2 \text{ and } (2, v-u+2)\cdot(2,u) = (u,v) \\ \cdots \\ p = u-1 \text{ and } (2, v-u+2)\cdot(u-1,u) = (u,v). \end{cases}$$

□

Corollary 2.1. *The set $R(z)$ of right divisors of an element $z = (u,v) \in M_{\lneqq}$ is finite and the number $\tau_r(u,v)$ of all right divisors of (u,v) is given by*

$$\tau_r(u,v) = |R(z)| = v - u + \frac{u(u-1)}{2}.$$

A Möbius monoid M is a decomposition-finite monoid (i.e. for any $s \in M$ there is a finite number of pairs $(t, t') \in M \times M$ such that $s = tt'$) in which the identity e is indecomposable, and $st = t$ implies $s = e$ for any $s, t \in M$. The monoid $M_{\lneqq}$ is decomposition-finite since $R(z)$ is finite for any $z \in M_{\lneqq}$. The identity $e = (1,2)$ is indecomposable (since $|R(e)| = 1$), and $M_{\lneqq}$ is right cancellative. Thus, we have

Proposition 2.2. *The non-commutative monoid $(M_{\lneqq}, \cdot)$ is a right cancellative Möbius monoid.*

Proposition 2.3. *The convolution $\xi * \eta$ of two arithmetic functions ξ and η on $M_{\lneqq}$ is given by:*
$(\xi * \eta)(u,v) =$

$$\sum_{i=2}^{v-u+1} \xi(1,i)\eta(u, v-i+2) + \sum_{j=2}^{u}[\xi(u-j+2, v-j+2)\sum_{k=1}^{j-1}\eta(k,j)],$$

where $\sum_{j=2}^{u}[\xi(u-j+2, v-j+2)\sum_{k=1}^{j-1}\eta(k,j)] = 0$ *if* $u = 1$.

3 The half-factorial monoid $M_{\lneqq}$

The study of half-factorial monoids is a main subject in non-unique factorization theory. In [6], Haukkanen and the author showed that a commutative Möbius monoid which arise from a combinatorial bisimple inverse monoid, satisfies a unique factorization theorem. For any monoid M with units $M^\times$ an element $s \in M - M^\times$ is called atom if for all $t, t' \in M$, $s = tt'$ implies $t \in M^\times$ or $t' \in M^\times$. The monoid M is said to be atomic if every $s \in M - M^\times$ is a product of finitely many atoms of M. A half-factorial monoid is an atomic monoid in which every two decompositions into atoms of a non-unit element s have the same length, denoted $\ell(s)$.

By virtue of Corollary 2.1, for $z \in M_{\lneqq}$ we have $|R(z)| = 2$ if and only if $z = (1,3)$ or $z = (2,3)$. Since $M_{\lneqq}{}^{\times}$ is a singleton it follows that in $M_{\lneqq}$ there are only two atoms:

$$a = (1,3) \text{ and } b = (2,3).$$

It is straightforward to see that

$$a^m = (1, m+2) \text{ (expansion)} \quad \text{and} \quad b^n = (n+1, n+2) \text{ (translation)}.$$

Since

$$ba = b^2,$$

it follows

Lemma 3.1. *For any positive integers m and n we have:*

(i) $(ba)^n = b^{2n}$;

(ii) $(ab)^n = ab^{2n-1}$;

(iii) $b^m a^n = b^{m+n}$;

(iv) $a^m b^n a^p b^q = a^m b^{n+p+q}$.

For notational convenience, the elements a, b and e will be often written as ab^0, a^0b and a^0b^0, respectively. Now, since $(u, u+i) = (1, i+1) \cdot (u, u+1) = a^{i-1}b^{u-1}$, it follows

Lemma 3.2. *For any $(u,v) \in M_{\lneqq}$ we have*

$$(u,v) = a^{v-u-1}b^{u-1}.$$

Using Lemma 3.2 it is easy to see that

Lemma 3.3. *We have:*

$$a^m b^n = (n+1, m+n+2).$$

By Lemma 3.2, every element $(u,v) \neq (1,2)$ can be expressed as a product of atoms. This representation is not unique (for example: $(3,4) = ba = b^2$). It is straightforward to check that every non-identity element (u,v) has a unique decomposition of the form $(u,v) = a^m b^n$, called the *normal representation* of (u,v). The assertions of Lemma 3.1 lead us to the following result: every two decompositions into atoms in $M_{\lneqq}$ of a non-identity element (u,v) have the same length. Thus we have (the part two of the result follows from Lemma 3.2):

Proposition 3.1. *The non-commutative, right cancellative Möbius monoid $M_{\lneqq}$ is half-factorial, and the length $\ell(u,v)$ of a non-identity element (u,v) is given by*

$$\ell(u,v) = v - 2.$$

4 The Möbius function

It is straightforward to check that the normal representation in $M_{\lneqq}$ of the product of two elements of $M_{\lneqq}$ is given by:

Lemma 4.1. *If $(u,v) = a^m b^n$ and $(u',v') = a^p b^q$ are the normal representations of the non-identity elements (u,v) and (u',v'), respectively, then*

$$(u,v)\cdot(u',v') = \begin{cases} a^{m+p}b^q & \text{if} \quad n=0 \\ a^m b^{n+p+q} & \text{if} \quad n>0 \end{cases}$$

is the normal representation of the product $(u,v)\cdot(u',v')$.

Now, Proposition 2.1, Corollary 2.1 and Proposition 2.3 (using Lemmas 3.2, 3.3 and 4.1) imply

Proposition 4.1. *(1) Let $z = a^m b^n \in M_{\lneqq}$. The set $R(z)$ of right divisors of z is the following one:*

$$R(z) = \begin{cases} \{a^m, a^{m-1}, ..., a, e, \} & \text{if} \quad n=0, \\ \{a^m b^n, a^{m-1}b^n, ..., ab^n, b^n\} \cup \{e\} \cup \{(a,b\} \cup \{a^2, ab, b^2\} \cup ... & \\ ... \cup \{a^{n-1}, a^{n-2}b, a^{n-3}b^2, ..., ab^{n-2}, b^{n-1}\} & \text{if} \quad n>0. \end{cases}$$

(2) the number $\tau_r(a^m b^n)$ of all right divisors of $z = a^m b^n \in M_{\lneqq}$ is given by

$$\tau_r(a^m b^n) = |R(z)| = m + 1 + \frac{n(n+1)}{2}.$$

(3) The convolution $\xi * \eta$ of two arithmetic functions ξ and η on $M_{\lneqq}$ is given by:

$$(\xi * \eta)(a^m b^n) = \sum_{i=0}^{m} \xi(a^{m-i})\eta(a^i b^n) + \sum_{j=2}^{n+1} [\xi(a^m b^{n-j+2}) \sum_{k=1}^{j-1} \eta(a^{j-k-1}b^{k-1})],$$

where $\sum_{j=2}^{n+1}[\xi(a^m b^{n-j+2})\sum_{k=1}^{j-1}\eta(a^{j-k-1}b^{k-1})] = 0$ if $n=0$.

Proposition 4.2. *The Möbius function μ of the Möbius monoid $M_{\lneqq}$ is given by*

$$\mu(a^m b^n) = \begin{cases} 1 & \text{if} \quad [m=0, n=0] \text{ or } [m=0, n=2]; \\ -1 & \text{if} \quad [m=1, n=0] \text{ or } [m=0, n=1] \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The above result can be obtained directly from the defining property of Möbius' μ-function: $\zeta * \mu = \delta$, where δ is the convolution identity (i.e., $\delta(e) = 1$, and $\delta(z) = 0$ if $z \neq e$). Thus we have

$$\sum_{i=0}^{m} \mu(a^i b^n) + \sum_{j=2}^{n+1}[\sum_{k=1}^{j-1} \mu(a^{j-k-1}b^{k-1})] = \begin{cases} 1 & \text{if} \quad m=n=0 \\ 0 & \text{otherwise}, \end{cases}$$

where $\sum_{j=2}^{n+1}[\sum_{k=1}^{j-1}\mu(a^{j-k-1}b^{k-1})] = 0$ if $n=0$. It follows:

(i) $m = n = 0 \Rightarrow \mu(a^0 b^0) = \mu(e) = 1$;

(ii) $m > 0, n = 0 \Rightarrow \sum_{i=0}^{m} \mu(a^i) = 0$;

(iii) $m > 0, n > 0 \Rightarrow 0 = \sum_{i=0}^{m} \mu(a^i b^n) + \sum_{j=2}^{n+1} [\sum_{k=1}^{j-1} \mu(a^{j-k-1} b^{k-1})] = \mu(a^m b^n) + \sum_{i=0}^{m-1} \mu(a^i b^n) + \sum_{j=2}^{n+1} [\sum_{k=1}^{j-1} \mu(a^{j-k-1} b^{k-1})] = \mu(a^m b^n)$.

(iv) $m = 0, n > 0 \Rightarrow \mu(b^n) + \sum_{j=2}^{n+1} [\sum_{k=1}^{j-1} \mu(a^{j-k-1} b^{k-1})] = 0$, and using (iii) we obtain: $\mu(b^n) + \sum_{i=0}^{n-1} [\mu(a^i) + \mu(b^i)] - \mu(1) = 0$.

Now, it is straightforward to see that:

(1) (i) and (ii) imply that $\mu(a^m b^n) = \begin{cases} 1 & \text{if} \quad m = n = 0 \\ -1 & \text{if} \quad m = 1, n = 0 \\ 0 & \text{if} \quad m > 1, n = 0, \end{cases}$

(2) (iii) says that $\mu(a^m b^n) = 0$ if $m > 0, n > 0$,

(3) (iv) implies that $\mu(a^m b^n) = \begin{cases} -1 & \text{if} \quad m = 0, n = 1 \\ 1 & \text{if} \quad m = 0, n = 2 \\ 0 & \text{if} \quad m = 0, n > 2, \end{cases}$

and the proof is complete. □

Using Lemma 3.3,

Corollary 4.1. *We have:*

$$\forall (u,v) \in M_{\lneq} : \quad \mu(u,v) = \begin{cases} 1 & \textit{if} \quad [u = 1, v = 2] \textit{ or } [u = 3, v = 4]; \\ -1 & \textit{if} \quad [u = 1, v = 3] \textit{ or } [u = 2, v = 3] \\ 0 & \qquad \textit{otherwise}. \end{cases}$$

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