Notes on Number Theory and Discrete Mathematics
ISSN 1310-5132
Vol. 21, 2015, No. 3, 1–21

"Theory of Numbers" of a complete region

Ranjit Biswas

Department Department of Computer Science
Jamia Hamdard University
Hamdard Nagar, New Delhi–110062, India
e-mail: `ranjitbiswas@yahoo.com`

Abstract: This is sequel to our previous work [2, 3, 6] on region algebra. An important contribution in [2] is that the minimum platform required and being used so far by the world for practicing elementary algebra is unearthed and uniquely identified which is not group, ring, field, module, linear space, algebra over a field, associative algebra over a field, division algebra, or any existing algebraic system in general, but 'Region Algebra'. The properties of region algebra are interesting as this is the minimal algebra which justifies free and fluent practice of elementary as well as higher algebra. This important identification was missing so far in any past literature of algebra or mathematics, and thus it is surely a unique algebra of absolute integrated nature. A new theory called by "Theory of Objects" and as a special case of it the classical "Theory of Numbers" were also studied in [2, 6]. In this paper we say that every complete region A has its own 'Theory of Numbers' called by 'Theory of *A*-numbers', where the classical 'Theory of Numbers' is the 'Theory of *RR*-numbers' corresponding to the particular complete region *RR*. For the sake of presentation and to avoid any confusion we consider three theories here but finally we arrive at a unified unique theory at the end. The three theories designated in this paper are: **Theory-1 (Theory of Numbers)** which is exactly the existing "Theory of Numbers" in the literature (on real numbers and complex numbers); **Theory-2 (Theory of Objects)** which is about combinatorics on Region Algebra, about prime objects and composite objects, about a new 'Theory of Numbers' corresponding to every complete region (viz. the 'Theory of *A*-numbers' is corresponding to the complete region *A*, etc.); and **Theory-3 (Theory of RR-numbers)** which is all about the "Theory of *RR*-numbers", a particular case of the "Theory of A-numbers" of Theory-2 where the region *A* is the complete region *RR*. In fact Theory-1 happens to be a special case of Theory-3, but initiating of Theory-3 done by the author is not with the purpose of 'Making a generalization of the Theory-1'. It may also be noted that the Theory-3 is a special case of "Theory of *A*-numbers" where "Theory of *A*-numbers" is derived from the "Theory of Objects" of Theory-2. The "Theory of Objects" also induces a new field called by "Object Geometry" of a complete region, being a generalization of our rich classical geometry of the existing notion. It is claimed that the "Theory of Objects" will play a huge role to the Number Theorists in a new direction.

Keywords: object, region, extended region, calculus space, complete region, 2-to-1 bijection, absolute partition, positive object, negative object, bachelor, conjugate bachelor, exact division, prime object, composite object, onteger, unit length, inverse unit length, natural A-number (of a complete region A), R_A value, object line, X_A-axis, Y_A-axis, object plane, object circle, compound number.
AMS Classification: 97G40, 97G99, 11F23, 11Gxx, 14Pxx, 14Qxx.

1 Introduction

This is sequel to our previous work [2, 3, 6] in which we introduced the notion of region algebra, theory of objects and calculus space. It has been justified in [2, 6] with the help of several examples that many of the simple results, formula, equalities, identities, rules etc. of elementary algebra are not valid, in general, in a group, ring, field, module, linear space, algebra over a field, associative algebra over a field, division algebra, or in any existing algebraic system [1, 9–13, 15], but in region algebra. It is unearthed and uniquely identified that the minimum platform required for practicing elementary algebra is the region algebra. It is also noticed that a region algebra is basically a "commutative division F-algebra", but the most important contribution in [2, 6] is that the huge unique potential of region algebra is discovered and consequently its characteristics are well studied, which validates the free and fluent practice of elementary as well as higher algebra (not possible by any other existing algebra, even not by division algebra) by the world scientists. Theory of region algebra is surely a major field, deserves major attention for understanding the existing huge volume of literature on algebra in an integrated way. In [3], we introduced the notion of extended region, 2-to-1 bijective mapping, positive and negative objects on making an absolute partition of a real region, object line of a region and then a calculus space on which a new calculus can be developed. The preliminaries about the region algebra are not discussed further in this work, and for details one could see [2].

In this paper we introduce two parts of region mathematics.

In the first part we introduce that for every complete region A, there is a corresponding Theory of Numbers called by "Theory of A-numbers". We then show that the existing classical 'Theory of Numbers' is a particular instance of our newly introduced notion of "Theory of A-numbers". For the sake of smooth presentation, we initially designate three theories here which finally integrate into a unified unique theory: **Theory-1 (Theory of Numbers)** which is exactly the <u>existing</u> "Theory of Numbers" in the literature; **Theory-2 (Theory of Objects)** which is about combinatorics on Region Algebra, about prime objects and composite objects, about a new 'Theory of Numbers' corresponding to every complete region (viz. the 'Theory of A-numbers' is corresponding to the complete region A, 'Theory of B-numbers' is corresponding to the complete region B, etc.); and **Theory-3 (Theory of RR-numbers)** which is all about the "Theory of RR-numbers", a particular case of the "Theory of A-numbers" of Theory-2 if the region A is the complete region RR.

In the second part we introduce another new field "Object Geometry" corresponding to a complete region A, and also we show that the existing classical 'Geometry' is a particular instance of our newly introduced "Object Geometry".

2 Region Algebra "Calculus Space" & "Complete Region"

It has been observed that there is a genuine need to introduce a new algebraic system having unique self-identity in order to provide a minimal but sufficient platform where 'elementary mathematics' can be fluently practiced with algebraic right and validity. This job is done in [2] by introducing the algebraic system called by "Region", which is a very simple algebraic system, very complete and sound.

Definition 2.1. Region. Consider a non-null set A equipped with three binary operations $\oplus$, * and $\bullet$ such that for a given field $(F, +, .)$, the following three conditions are satisfied : -

(i) $(A, \oplus, *)$ forms a field,

(ii) $(A, \oplus, \bullet)$ forms a linear space over the field $(F, +, .)$, and

(iii) A satisfies the property of : "Compatibility with the scalars of the field F"

i.e. $(a\bullet x) * (b\bullet y) = (a.b)\bullet (x*y)\ \forall a, b\in F$ and $\forall x, y\in A$. Then the algebraic system $(A, \oplus, *, \bullet)$ is called a **Region** over the field $(F, +, .)$. If there is no confusion, we may simply use the notation A to represent the region $(A, \oplus, *, \bullet)$, for brevity.

In the region $(A, \oplus, *, \bullet)$, its component algebraic system $(A, \oplus, *)$ is a field. Thus we see that the region A is a commutative division algebra. Also the other component algebraic system $(A, \oplus, \bullet)$ is a linear space over the field F. Considering the distributive properties of the field $(A, \oplus, *)$ along with the condition (iii) of the definition 2.1, it can be observed that the region A is F-algebra. Thus a region is a **"Commutative Division *F*-Algebra"**, but defined independently and uniquely with a self-identity in [2] with its important properties and results.
Consider any region A. A variable x which can take values from the region A is called an 'object variable'. The concept of positive objects and negative objects of a complete region is introduced in [3]. In this section we review the results and make further analysis.

Definition 2.2. Extended Region. Consider a region A. If we include two more objects $+\propto_A$ and $-\propto_A$ in A, where $+\propto_A = \frac{x}{0_A}$ where x $(\neq 0_A)$ is any positive object, and $-\propto_A = \frac{z}{0_A}$ where z $(\neq 0_A)$ is any negative object, then the set $A \cup \{+\propto_A, -\propto_A\}$ is called to be an extended region. Note that an extended region is not a region. But if we say that A is an extended region, it implies that A is a region and two infinities are included to it.

Definition 2.3. 2-to-1 Bijective Mapping
Consider two non-null sets X and Y. A function $f: X \to Y$ is said to be a '2-to-1 Bijective Mapping' if
(i) f is onto, and
(ii) $\forall y \in Y$, $\exists$ two and only two distinct (not same) elements x_1 and x_2 in X such that $f(x_1) = y = f(x_2)$.

For example, the function $f: R - \{0\} \to R^+$ given by $f(x) = x^2$ is a 2-to-1 Bijective Mapping.

In [3], we explored the 'calculus space' as the minimum necessary requirements of mathematics if we want to develop a new calculus.

Definition 2.4. Calculus Space. Consider any real region $A = (A, \oplus, *, \bullet)$ over the field $(R, +, .)$. Then A forms a Calculus Space if the following conditions are true:

(i) A is an extended real region.

(ii) A is a normed complete metric space with respect to a norm $\|.\|$ and the corresponding induced metric $\rho(x, y) = \|x \sim y\|$, (i.e. $\|x\| = \rho(x, 0_A)$).

(iii) The norm $\|.\|$ is 2-to-1 bijective mapping from $A - \{0_A\}$ to R^+.

(iv) A is a chain w.r.t. a total order relation '$\leq$' .

As a particular instance, if we choose the real region A to be the RR region and $\|x\| = |x|$ in RR where $\rho(x, y) = \|x-y\| = |x-y|$ and the *RR* region is a chain w.r.t. the crisp order relation "$\leq$", then the corresponding calculus happens to be the classical calculus (developed independently by Newton and Leibniz). The set R of real numbers is so interesting that it very comfortably forms the region *RR*, and the region *RR* is so beautiful that it satisfies all the above four conditions to form a Calculus Space (an eligible platform on which a calculus can be developed). Consequently, it is clear now that the classical calculus developed independently by Newton and Leibniz happens to be on the calculus space *RR*.

The following facts [9, 15, 16] may be recalled that the metric associated with this norm, i.e. the metric $\rho(x, y) = \|x \sim y\|$ has the following special properties :

(i) **'Translation Invariance'**, i.e. $\forall z \in A$ we have $\rho(x \oplus z, y \oplus z) = \rho(x, y) = \|x \sim y\|$, and

(ii) **'Homogeniety'**, i.e. $\forall r \in R$ we have $\rho(r \bullet x, r \bullet y) = |r|.\|x \sim y\| = |r|.\rho(x, y)$.

We already have a complete idea about the development and growth of the classical calculus since its inception (happened to be developed on the calculus space *RR*). In an analogous way, the basic concepts of any new calculus (new differential calculus) viz. limit, continuity, differentiability of a function of objects, etc. are explained in [3] on a general calculus space.

Definition 2.5. Complete Region. A real region which forms a calculus space satisfying the above properties is called a **"complete region"**. For instance, the region *RR* is a complete region.

The collection of all complete regions is called the **region universe $\boldsymbol{\Sigma}$**.

Definition 2.6. 'Absolute Partition' of the complete region A. We defined 'Absolute Partition' of the region $A = (A, \oplus, *, \bullet)$ in [3]. Consider a partition P_A of a region A (forming calculus space) into three mutually disjoint non-null sets A^+, A^- and $\{0_A\}$ such that

(i) $A^+ = \{a : a \in A \text{ and } 0_A < a\}$

(ii) $A^- = \{a : a \in A \text{ and } a < 0_A\}$.

Clearly, $\forall a \in A^+, \sim a \in A^-$ and $\forall b \in A^-, \sim b \in A^+$.

(Note: we say that $a < b$ iff $a \leq b$ and $a \neq b$ where "$\leq$" is the total order relation of the chain A).

This partition P_A, once made, must be regarded as an 'absolute partition' for the region A over which one desires to develop a calculus and any branch of region mathematics in any direction. It is called to be absolute in the sense that it generates the sign of every object of A, positive or negative, which will remain absolute for the complete literature of the corresponding calculus or corresponding region mathematics. The elements of A^+ are said to be positive objects and the elements of A^- are said to be negative objects. The object 0_A is neither in A^+ nor in A^-, and so we say that 0_A is neither a positive object nor a negative object. The attribute of being positive or negative is called the sign of the object, and 0_A is not considered to have a sign.

For the sake of avoiding confusion, let us designate the classical 'Theory of Numbers' (available in the existing literature) by **"Theory-1".** In the next section we discuss about a new theory called by "Theory of Objects" which we designate by "**Theory-2".** However, in Theory-2 we will have a special interest on the "Theory of *RR*-numbers" of it, which we designate as **"Theory-3".** <u>But these are temporary designations and will be ignored while we conclude the work here at the end.</u>

3 Theory of Objects (Theory-2)

By an object we mean an element of a region. In this section we develop a new theory called by "Theory of Objects" in Region Algebra. As a part of it, we first of all discuss about a new Theory of Numbers and then about Object Geometry.

Consider a complete region A. Corresponding to every complete region A, we derive a corresponding 'Theory of Numbers' called by "Theory of A-numbers". If RR, A, B, C, … are the complete regions in region algebra, then the corresponding theories are "Theory of RR-numbers", "Theory of A-numbers", "Theory of B-numbers", "Theory of C-numbers", ... etc. as shown in Figure 1.

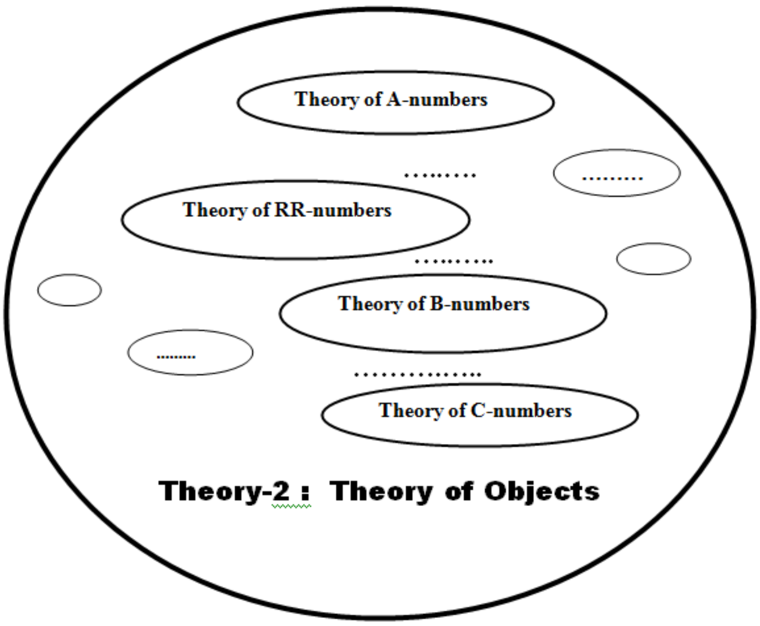

Figure 1. Every complete region has its own Theory of Numbers

3.1 Theory of *A*-numbers

3.1.1 Object line in a complete region *A*

Consider a complete region A. A line can be drawn on which one point may be fixed to be the location for the object 0_A, with all positive objects of A having location to the right and all negative objects of A having location to the left of 0_A. Thus the 'positive direction' of the line can be called to be X_A-axis and the 'negative direction' of the line can be called to be $X_A{}^1$-axis. And the line which the objects of the region A is considered to lie upon is called the **Object Line for the complete region** ***A*** (see Figure 2 and Figure 3).

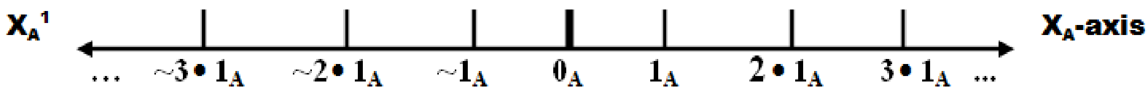

Figure 2. Object line of the region A with consecutive equi-spaced object points

The term 'equi-spaced' in the caption of Figure 2 is well understood in the sense of the corresponding metric (or norm) of the region A, i.e. for any real integer r, $\rho(r \bullet 1_A, (r+1) \bullet 1_A)$ = constant (independent of r), in the Theory of A-numbers.

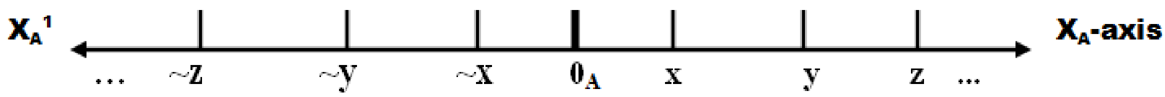

Figure 3. Objects line of the region A, a general view

Since $A = (A, \oplus, *, \bullet)$ is complete (normed complete metric space), there are no "points missing" from it (inside or at the boundary). Since A is a chain, every object has a unique address on this linear continuum $X_A{}^1X_A$; and corresponding to every address (point) on this linear continuum $X_A{}^1X_A$ there is a unique object of A.

3.1.2 Unit length & inverse unit length in a complete region *A*

Consider a complete region $A = (A, \oplus, *, \bullet)$. For $x_A \in A$, we use the notation x_a to denote $\|x_A\| = x_a$ which is a positive real number. If x_A is a positive object on the object line, then the distance of x_A from the point O (the location of the object 0_A on the linear continuum $X_A{}^1X_A$) is denoted by x_a which is a positive real number (we use the convention that $\sim x_A$ is at a distance of $-x_a$ from the point O).

Corresponding to the unit element 1_A of the complete region A, the positive real number 1_a (i.e. $\|1_A\|$) is called the **'unit length'** in the Theory of A-numbers. Clearly $0_a = 0$, and it may also be noted here that in general $1_a \neq 1$ (where 0 is the o_{RR} and 1 is the 1_{RR}). Suppose that $1/1_a = ə_a$. The positive real number $ə_a$ is called **'Inverse Unit Length'** in *A*.

Clearly $1_a . ə_a = 1$ and in general $1_a \neq ə_a$ for *A*. However, for the particular complete region RR we have $1_{rr} = ə_{rr} = 1$ ($= 1_{RR}$) .

3.1.3 'Ontegers' in the complete region *A*

Consider the object x_A in the complete region *A*. Consider the real number $x_a/1_a$ i.e. $x_a . ə_a$ which let us denote by the symbol x. Thus $x = x_a/1_a = x_a . ə_a$, which means $x_a = x.1_a \ \forall x_A \in A$. It may be noted here that in general $1_a \neq 1$, in a similar way $x_a \neq x$. However for a particular instance of the complete region RR, we have $x_{rr} = x$ ($= x_{RR}$).

If *m* is a real integer, then the object m_A is called an 'object integer' or **'onteger'** in the Theory of *A*-numbers.

Thus the ontegers in the Theory of *A*-numbers are 0_A, $\oplus 1_A$, $\sim 1_A$, $\oplus 2_A$, $\sim 2_A$, $\oplus 3_A$, $\sim 3_A$, ..., etc. The ontegers $\oplus 1_A$, $\oplus 2_A$, $\oplus 3_A$, $\oplus 4_A$, …, etc., are **'positive ontegers'** and the ontegers $\sim 1_A$, $\sim 2_A$, $\sim 3_A$, $\sim 4_A$, …, etc., are **'negative ontegers'**.

It is to be carefully noted that corresponding to any onteger $\oplus m_A$ of the complete region *A*, the distance m_a from the point o_A on the object line is a real number but need not necessarily is a real integer; and similarly corresponding to any onteger $\sim m_A$, the distance $-m_a$ is a real number but not necessarily is a real integer.

Corresponding to every complete region *A*, there is a "Theory of A-numbers". Consider the complete regions *RR*, *A*, *B*, *C*, *D*, … etc. and the corresponding "Theory of *RR*-numbers", "Theory of *A*-numbers", "Theory of *B*-numbers", "Theory of *C*-numbers", "Theory of *D*-numbers"… respectively. If we imagine a common object line for these different complete regions *RR*, *A*, *B*, *C*, *D*, …, etc., with the respective zero elements 0, 0_A, 0_B, 0_C, 0_D, …, being situated at exactly the same point on the common object line, then it is obvious that the respective unit elements $1(1_{RR})$, 1_A, 1_B, 1_C, 1_D, …, etc., will be situated in general at different points on the common line because of the fact that the 'unit length' is region-dependent. Thus, for any real number x, in general the points x, x_A, x_B, x_C, x_D, …, etc., will be situated at different locations on the common object line. Distance (if measured in a common scale, say with the help of real numbers) between two consecutive ontegers for any given complete region *A* on the object line will be same, but will be different for different complete regions. On the *RR* region line i.e. on the real number line, distance of the object $\oplus 1_{RR}$ or $\sim 1_{RR}$ from the object 0_{RR} (i.e. distance of the real number +1 or -1 from the number 0) is of unit length called us by 'one'. It may be noted that for every $x_A \in A$, x_a is in *R*.

It may also happen that the integer 1 of the set *R* (i.e. the onteger 1_{RR} of the Theory of *RR*-numbers) is not an onteger in the "Theory of *A*-numbers" and the onteger 1_a of the "Theory of *A*-numbers" is not an onteger in the "Theory of *RR*-numbers" (i.e. is not an integer in the classical "Theory of numbers"). The main source of such differences lies in the difference of size of 'unit length' of different complete regions.

Thus "Theory of *A*-numbers" is different for different complete region *A* in the Theory of Objects, whereas the classical "Theory of Numbers" being available in the existing literature and being practiced by us traditionally so far is a content of the "Theory of *RR*-numbers" in the Theory of Objects.

The following proposition is straightforward and quite important.

Proposition 3.1. Corresponding to a real number x ($-x$), there is a unique object $\oplus x_A$ ($\sim x_A$) in the complete region A and hence a unique corresponding real number x_a ($-x_a$).

Definition 3.1 'R_A value' of a real number x. Let A be a complete region. Consider the 1-to-1 mapping $R_A : R \rightarrow R$ defined by $R_A(x) = x_a \;\; \forall x \in R$. Then the real number x_a is called the '**R_A value**' of x denoted by $R_A(x) = x_a$ corresponding to the complete region A. Clearly, in that case $R_A(-x) = -x_a$. Also $R_A(0) = 0_a$, and $R_A(1) = 1_a$.

It is obvious that $R_{RR} : R \rightarrow R$ is an identity mapping.

Definition 3.2 'Set of R values' and 'Set of R objects' corresponding to a real number x. If RR, A, B, C, D, … are the complete regions in the region universe $\boldsymbol{\Sigma}$, then for any given real number x the set $\boldsymbol{\Sigma}_{\boldsymbol{x}} = \{x_{rr} (= x), x_a, x_b, x_c, x_d, \ldots\}$ is called the '**Set of R values**' of x in the region universe $\boldsymbol{\Sigma}$ and the set $\boldsymbol{\Sigma}_{\mathbf{X}} = \{x_{RR} (= x), x_A, x_B, x_C, x_D, \ldots\}$ is called the '**Set of R objects**' of x in the region universe $\boldsymbol{\Sigma}$**.** Although we call $\boldsymbol{\Sigma}_{\boldsymbol{x}}$ a set, it could be a multiset (bag) too. Collection of R values of the real number 1 is the set (multiset) of all unit length values forming $\boldsymbol{\Sigma}_{\mathbf{1}}$, and the Collection of R values of the real number 0 is the set (multiset) $\boldsymbol{\Sigma}_{\mathbf{0}}$**.**

Definition 3.3 Natural A-ontegers and Natural A-numbers. In the Theory of A-numbers, the positive ontegers $\oplus 1_A$, $\oplus 2_A$, $\oplus 3_A$, $\oplus 4_A$, … are called the **Natural A-ontegers** and the positive real numbers 1_a, 2_a, 3_a, 4_a, … are called the **Natural A-numbers**.

For instance, in the Theory of RR-numbers, the Natural RR-ontegers are 1_{RR}, 2_{RR}, 3_{RR}, 4_{RR}, … and the Natural RR-numbers are 1_{rr}, 2_{rr}, 3_{rr}, 4_{rr}, … **.** Here the Natural RR-ontegers and Natural RR-numbers are same numbers i.e. the classical natural numbers 1, 2, 3, 4, … .

Let us consider three complete regions RR, A and B (say). The natural RR-numbers (i.e., the classical natural numbers), natural A-numbers, natural B-numbers are as shown in Figure 4. The consecutive natural RR-numbers are equi-spaced on X_{RR}-axis, and same is true for consecutive natural A-numbers on X_A-axis, consecutive natural B-numbers on X_B-axis. But the unit lengths are different for different object lines corresponding to different complete regions.

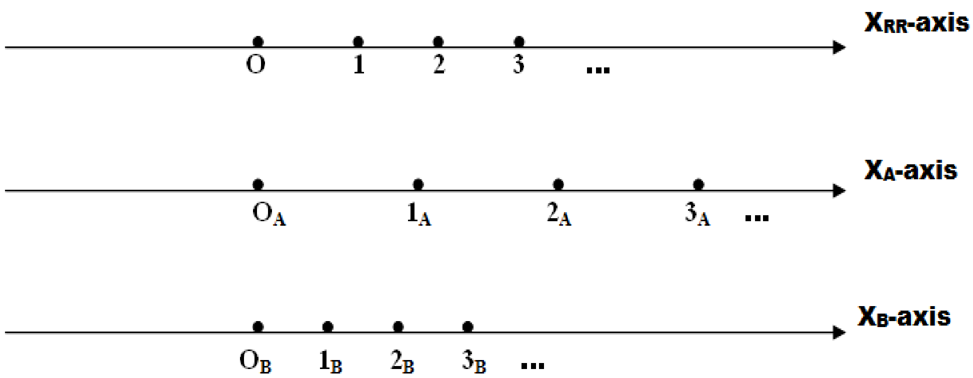

Figure 4. Natural Ontegers for "Theory of RR-numbers", "Theory of A-numbers" and "Theory of B-numbers" (a comparative view)

Consider the positive real number $x \in R$, and any three complete regions say RR, A and B. The corresponding three objects of Σ_X : x_{RR} (i.e., x itself) on the X_{RR}-axis, x_A on the X_A-axis, and x_B on the X_B-axis are shown in Figure 5. On their respective axis of linear continuum, the object x is at a distance x from O (i.e. O_{RR}), the object x_A is at a distance x_a from O_A and the object x_B is at a distance x_b from O_B. Here $x_{RR} = x.1_{RR}$ (where $x_{RR} = x$ and $1_{RR} = 1$), $x_a = x.1_a$, and $x_b = x.1_b$. The three real numbers 1_a (the metric distance of the unit object 1_A from the center point O_A), 1_b (the metric distance of the unit object 1_B from the center point O_B) and 1_{RR} or 1 (the metric distance of the unit object 1_{RR} from the center point O_{RR}) are not equal in general, a hypothetical case is shown in the Figure 4, where $1_b < 1_{RR}$ (=1) < 1_a.

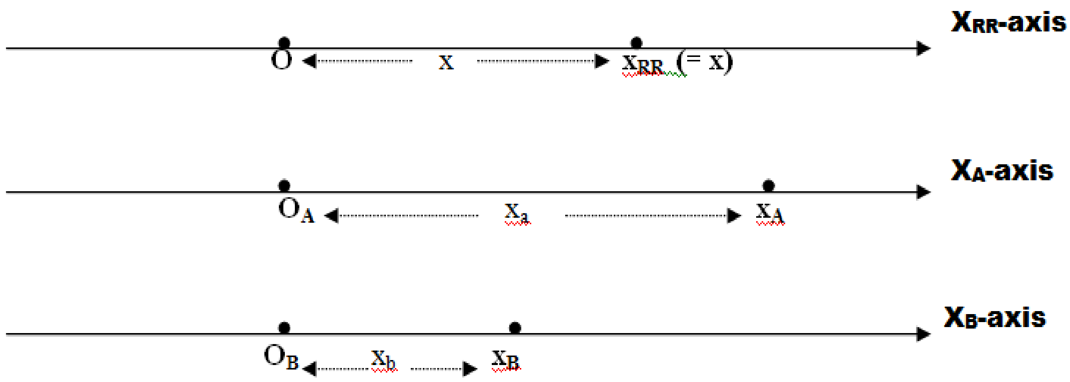

Figure 5. Three elements of Σ_X corresponding to "Theory of RR-numbers", "Theory of A-numbers" and "Theory of B-numbers" (a comparative view)

4 'Prime Objects' in the Theory of Objects

In this section we reproduce the notion of 'prime object' and 'composite object' in the Theory of Objects and few results on them. For this, first of all we consider the notion of 'bachelor set set' in a region.

Definition 4.1 'Bachelor set Set' in a Region. Let A be a region. A subset B of the region A is called a **'bachelor set set'** in A if

(i) $1_A \in B$, $0_A \notin B$ and

(ii) $\forall\ x (\neq 1_A) \in B$, $x^{-1} \notin B$.

Clearly, a bachelor set can never be a null set because the smallest bachelor set in a region A is the singleton $\{1_A\}$. Also, the self-inverse objects other than 1_A (like x where $x^2 = 1_A$) of the region A are not the members of any bachelor set of A. Any subset S of a bachelor set B in the region A is also a bachelor set in A if $1_A \in S$.

If B is a bachelor set in a region A, then the set

$$\widetilde{B} = \{\, y : y = x^{-1} \text{ where } x \in B \,\}$$

is also a bachelor set in A. This set $\widetilde{B}$ is called the **'conjugate bachelor'** of the bachelor set B in the region A.

Clearly, conjugate of the conjugate of B is B itself. The union of two bachelors in A need not be a bachelor set in A, but the intersection of two bachelors will be a bachelor set in A.

For every bachelor set B in A, $B \cap \widetilde{B} = \{1_A\}$.

If B and C are two bachelor set sets in the region A, then the conjugate of $(B \cap C)$ = $\widetilde{B} \cap \widetilde{C}$.

If $B = \widetilde{B}$, then the only case is that $B = \widetilde{B} = \{1_A\}$.

Example 4.1. Consider the region RR. Clearly the set N of natural numbers is a bachelor set in the region RR. The set $M = \{1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, \ldots\} = \{\, m : m = 1/n,\ n \in N$, where N is the set of natural numbers$\}$ is a bachelor set in the region RR. The set $L = \{1, 78.261, 9287, 83.5\}$ is also a bachelor set in the region RR. However the set R^+ of all positive real numbers is not a bachelor set in the region RR.

Proposition 4.1. If B of cardinality n is a bachelor set in the region A, then B has 2^{n-1} number of distinct sub-bachelors.

Proof: For $n = 1$, the result is true because the only possibility is that $B = \{1_A\}$.

Now consider the case $n > 1$. The two trivial sub-bachelors are $\{1_A\}$ and B. The cardinality of the set $B - \{1_A\}$ is $(n - 1)$, which is having 2^{n-1} number of subsets including the null set and the set $B - \{1_A\}$ itself. Adding the common element 1_A to each of these 2^{n-1} subsets will create 2^{n-1} number of bachelor set sets of A, being all the sub-bachelors of B. □

The division operations in a region are defined in [2]. We explain the operation of **'Exact Division'** of an element of a bachelor set B by another element of B [2].

Definition 4.2 'Exact Division' in a bachelor set set. Let B be a bachelor set in the region A. Consider two objects $x, y \in B$. We say that the object x exactly divides the object y in B, denoted by the notation "$x \,|_B\, y$", if $\exists z \in B$ such that $\frac{y}{x} = z$ in the region A. (Division of type $\frac{y}{x}$ has been defined in details while making characterizations of regions in [2]).

The notation "$|_B$" signifies the operation of 'exact division' in B, and the notation "$\nmid_B$" signifies the operation "cannot exactly divide" in B.

The following proposition is straightforward.

Proposition 4.2

(i) $x \,|_B\, x$ and $1_A \,|_B\, x$, $\forall x \in B$.

(ii) For $x \neq y$, if $x \,|_B\, y$, then $y \nmid_B x$, where $x, y \in B$.

Proposition 4.3. It may happen that for a given pair of objects x, y in a bachelor set B, neither $x\,|_B\,y$ nor $y\,|_B\,x$.

Proof: Consider a bachelor set C where x, y are in C and $x\,|_C\,y$ (such that $\frac{y}{x} = z$).

Now consider the bachelor set $B = C - \{z\}$. Clearly, x, y are in a bachelor set B but neither $x\,|_B\,y$ nor $y\,|_B\,x$. □

In the next subsection, we reproduce the notion of 'Composite Objects' and 'Prime Objects' in a bachelor set B of a region A.

Definition 4.3. 'Composite Object' in a bachelor set set. Let B be a bachelor set of a region A. An object $x \in B$ is called a 'Composite Object' in B, if $\exists\ p, q \in B - \{1_A\}$ such that $x = p * q$ in A.

Definition 4.4. 'Prime Object' in a bachelor set set. An object $x \in B - \{1_A\}$ is called a 'Prime Object' in B if x is not a composite object in B.

By construction here, there is no reason to check whether the element 0_A and the self-inverse elements (other than 1_A) of the region A are 'prime' or 'composite' or 'neither prime nor composite' in any bachelor set in the region A, as they cannot be members of any bachelor set in A. However, 1_A is the **only** element in any bachelor set B which is neither a prime object nor a composite object. For every other object x (i.e. if $x \neq 1_A$) in B, x is either a prime or a composite. Thus the following proposition is straightforward.

Proposition 4.4. There cannot be any object x in the bachelor set B in the region A which is both prime and composite.

If may be noted here that an object x may be prime in a bachelor set B of a region A, but may not be so in another bachelor set C of the same region A, where $x \in B$, C both.

Thus, for a given region, the property of prime, composite and 'neither prime nor composite' is dependent upon the concerned bachelor set set, and they must be members of this bachelor set set. For a given bachelor set set, checking an object of a region whether prime or composite or 'neither prime nor composite' where the object is not a member of the bachelor set set, is an invalid issue.

For a bachelor set V in a region A, the partition of the set V into three subsets: the set of Prime objects in V, the set of Composite objects in V, and the set of neither Prime nor Composite objects in V, is shown in Figure 6.

The following proposition is straightforward.

Proposition 4.5. If x is a prime (composite) object in a bachelor set B of a region R, then x^{-1} is a prime (composite) object in the conjugate bachelor set $\tilde{B}$, and conversely.

We present below examples of the notion of prime objects and composite objects in a bachelor set in a region.

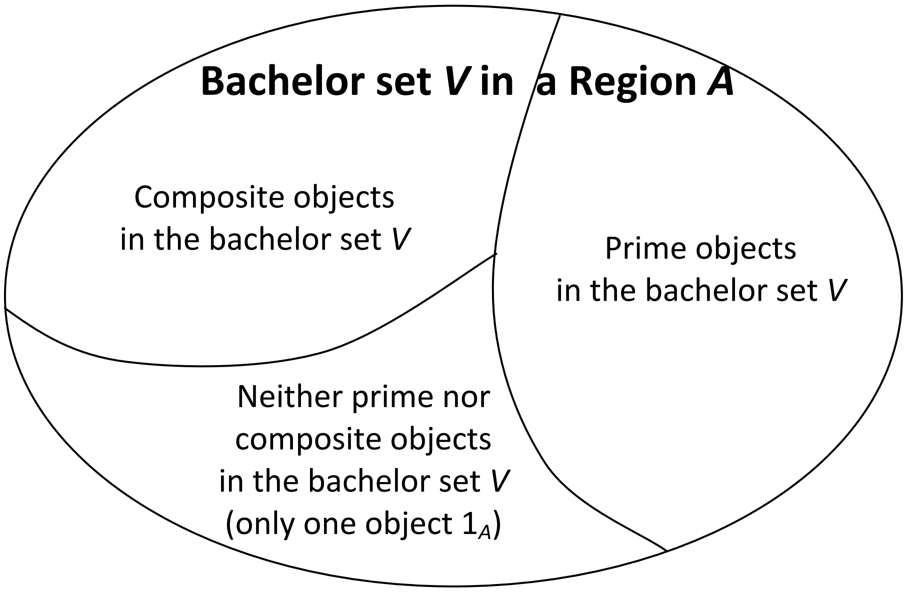

Figure 6. Prime, Composite and 'Neither prime nor composite' objects in a bachelor set *V* in the region *A*

Example 4.2. Consider the region *RR*. Consider the bachelor set *N* of the region *RR*, where *N* = {1, 2, 3, 4, 5, 6, 7, 8, …} = the set of natural numbers.

Clearly, the members 4, 6, 8, 9, 10, … are composite objects of the bachelor set *N* here in the region *RR*; and the members 2, 3, 5, … are prime objects of the bachelor set *N* in *RR* (which are popularly known as 'composite numbers' and 'prime numbers' respectively in the existing literature of the classical 'Theory of Numbers'). And 1 is the only object in the bachelor set *N* which is neither a prime object nor a composite object. There is no object in the bachelor set *N* which is both prime and composite (Proposition 4.4).

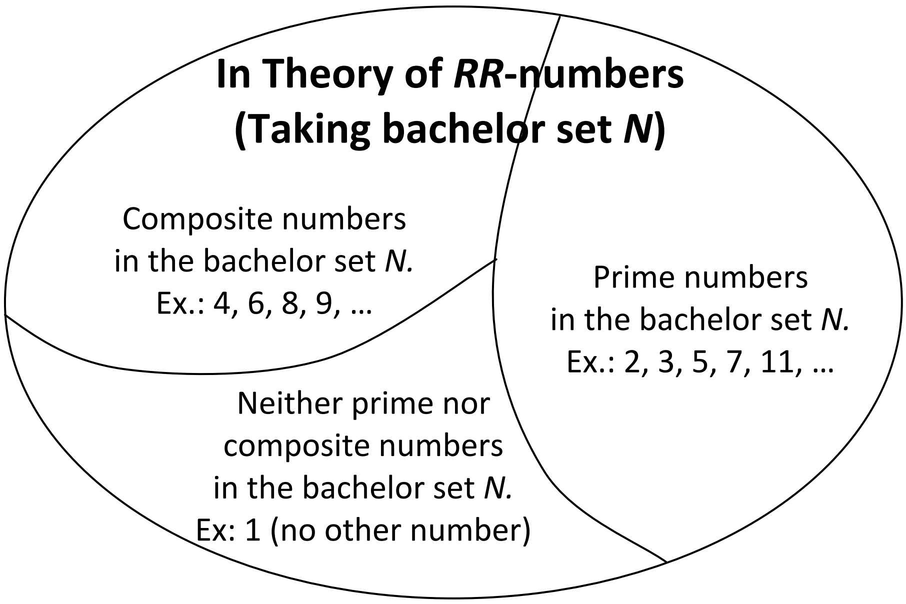

Figure 7. Prime, Composite and 'Neither prime nor composite' numbers in the bachelor set *N* (of natural numbers) in the region *RR*

Another example of prime and composite objects is given below.

Example 4.3. Consider the region *RR*. Consider the bachelor set *M* of the region *RR* where $M = \{1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, \ldots\} = \{ m : m = 1/n, n \in N$, where *N* is the set of natural numbers$\}$. Clearly, the members 1/4, 1/6, 1/8, 1/9, 1/10, … are composite objects of the bachelor set M here in the region RR; and the members 1/2, 1/3, 1/5, … are prime objects of *M* in *RR*. And 1 is the only object in the bachelor set *M*, which is neither a prime object, nor a composite object. There is no object in the bachelor set *M* which is both prime and composite (Proposition 4.4).

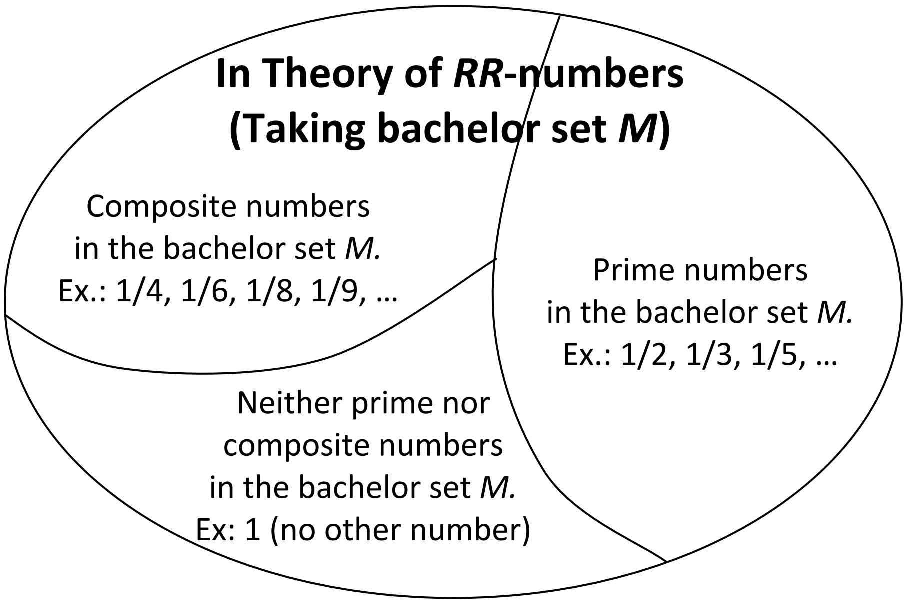

Figure 8. Prime, Composite and 'neither prime nor composite' numbers in the bachelor set *M* in the region *RR*

Example 4.4. Consider the bachelor set $L = \{1, 78.261, 9287, 83.5\}$ of the region *RR*. Clearly, the members 78.261, 9287, 83.5 are prime objects in the bachelor set *L*; there does not exist any composite object in *L*. And 1 is the only object in the bachelor set *L*, which is neither prime object nor composite object. There is no object in the bachelor set *L*, which is both prime and composite (Proposition 4.4).

Thus Example 4.2 above shows that the classical prime numbers (in the classical 'Theory of Numbers') are particular case of prime objects in the "Theory of *RR*-numbers" with bachelor set *N*, where "Theory of *RR*-numbers" is one of the topics under the subject of the "Theory of Objects".

Once the notion of "Theory of *A*-numbers" of a complete region *A* is developed, we are now in a position to initiate a corresponding 'Geometry' on the complete region *A*.

5 “Object Geometry” of a complete region A

In the Theory of Objects, for developing a new geometry called by “Object Geometry”, be it in a two dimensional object coordinate system, or in an n-dimensional object coordinate system, at least one complete region $A = (A, \oplus, *, \bullet)$ is required. Consider the object line and the X_A-axis corresponding to the complete region A. Consider a point x_A (a positive object) on the X_A-axis. Then for the infinitesimal small positive object Δx_A, the point $(x_A + \Delta x_A)$ will be at a distance $\|\Delta x_A\|$ from the point x_A along the positive direction of X_A-axis and the point $(x_A - \Delta x_A)$ will be at a distance $\|\Delta x_A\|$ from the point x_A along the negative direction X_A^1-axis. In the next section, we introduce “Y_A-axis” and, hence, a coordinate object plane in the Theory of Objects.

5.1 The Coordinate Plane of complete region $A = (A, \oplus, *, \bullet)$

We introduce first of all 2-D object geometry. It is a system of geometry where the position of points on the plane is described using an ordered pair of objects. A plane is a flat surface that goes on forever in both directions. If we were to place a point on the plane, object coordinate geometry gives us a way to describe exactly where it is by using two objects. Points are placed on the "object coordinate plane" as shown below in Figure 9. It has two scales – one running across the plane called the "X_A-axis" and another at right angles to it called the “Y_A-axis”. The point where the two axes cross is called the **origin** at which both x_A and y_A are 0_A. On the X_A-axis, as explained earlier that objects to the right of origin are positive and those to the left are negative. On the Y_A-axis, objects above the origin are positive and those below are negative.

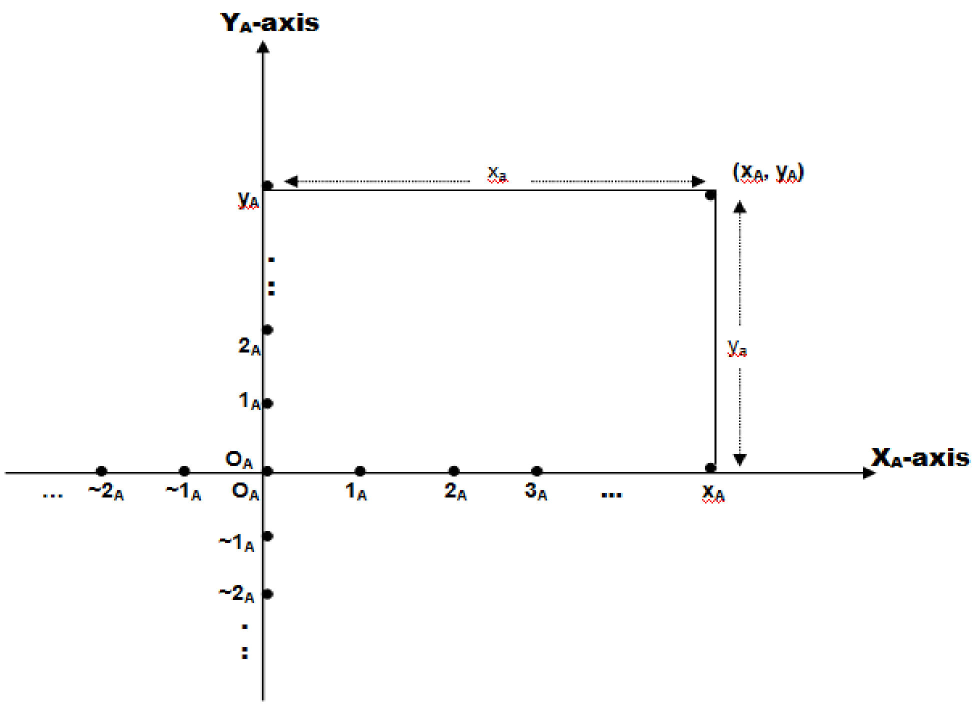

Figure 9. Objects coordinates on object plane of the region A

A point's location on the plane is given by two objects in the form of object coordinates (x_A, y_A), the first coordinate reveals where it is away from the Y_A-axis at parallel to the X_A-axis and the second coordinate reveals where it is away from the x_A-axis at parallel to the Y_A-axis (see Figure 9 above). There are four quadrants and sign convention rule is same as that of classical coordinate geometry, i.e. same for all the object geometry of all the complete regions.

5.2 Slope of an Object Line on the Object Plane

Slope of an object line passing through the two object points $P(x_{1A}, y_{1A})$ and $Q(x_{2A}, y_{2A})$ is the real number m_a given by (as shown in Figure 10):

$$
\begin{aligned}
m_a = \tan\theta \ &= (y_{2a} - y_{1a})/\ (x_{2a} - x_{1a}) \\
&= (y_2.1_a - y_1.1_a)/\ (x_2.1_a - x_1.1_a) \\
&= (y_2 - y_1)/\ (x_2 - x_1)
\end{aligned}
$$

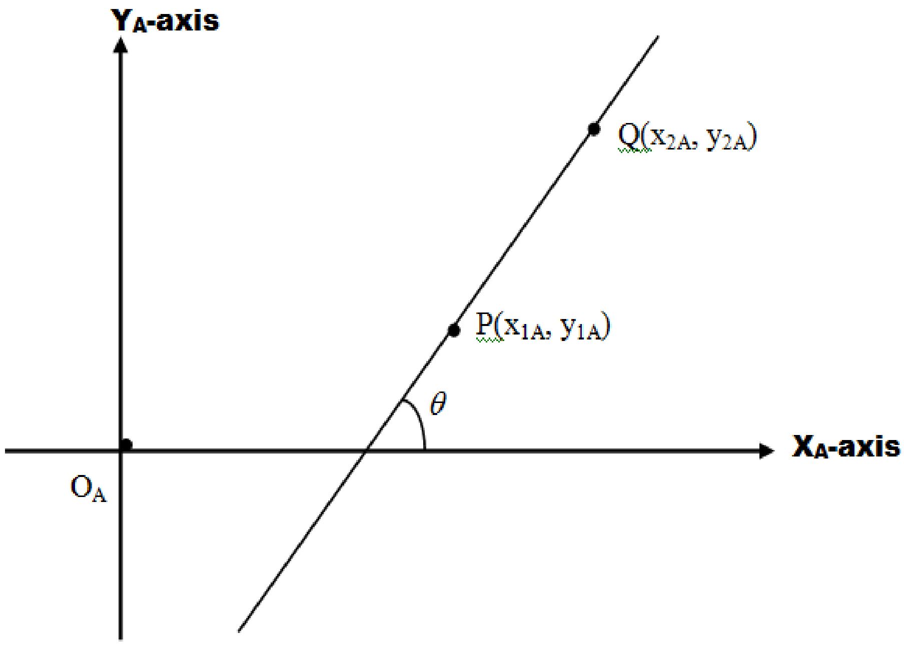

Figure 10. Slope of an objects line

This implies that slope of a line does not depend on the 'unit length' of the region. it is an absolute quantity irrespective of the region on which the object plane is drawn. Thus, slope of a line is region-independent.

Proposition 5.1. Pythagoras Theorem is valid in every Object Geometry.
Proof: Consider the Object Geometry corresponding to the complete region A. Let PQR be a right angled triangle (the angle PQR being the right angle) on the object plane of the complete region A (Figure 11(a)).

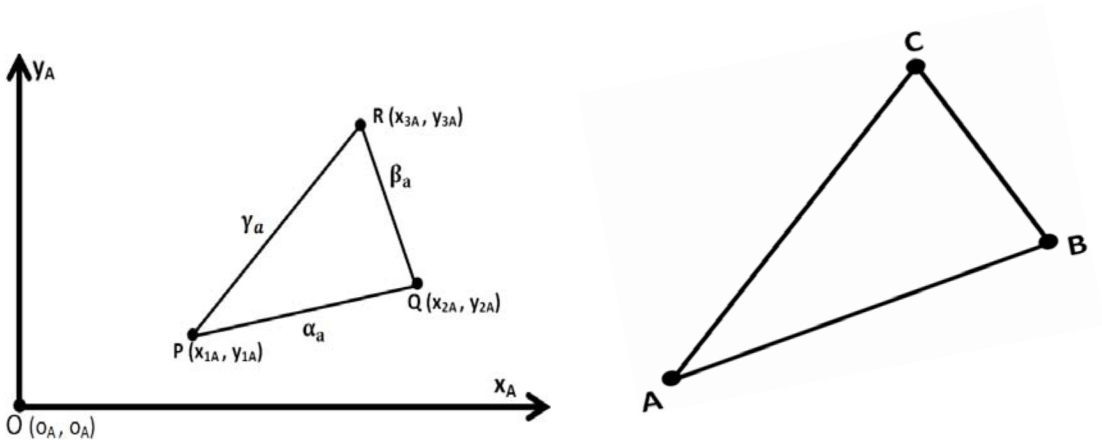

Figure 11(a), (b). Right angled triangles in two object planes

Now, using the homogeneity property of the metric $\rho\,(x, y) = \|x \sim y\|$, we can find a right-angled triangle ABC (in fact there are infinite numbers of such triangles) on the real coordinate plane i.e. on the object plane of RR region, where

$$\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC} = 1_a \tag{1}$$

Since slope of a line is region-independent, the right-angled property of the classical triangle *ABC* is guaranteed (the angle *ABC* being the right angle, see Figure 11(b)) on the coordinate plane from the right-angled property of the triangle *PQR*. Since Pythagoras theorem is valid in the triangle *ABC*, it is also so in the triangle *PQR* using equation (1). □

5.3 Distance between two object points on the object plane

Consider the two object points $P(x_{1A}, y_{1A})$ and $Q(x_{2A}, y_{2A})$ on the object plane (see Figure 12). Distance *PQ* is the positive real number r_a where

$$r_a = \{(y_{2a} - y_{1a})^2 + (x_{2a} - x_{1a})^2\}^{1/2}\,.$$

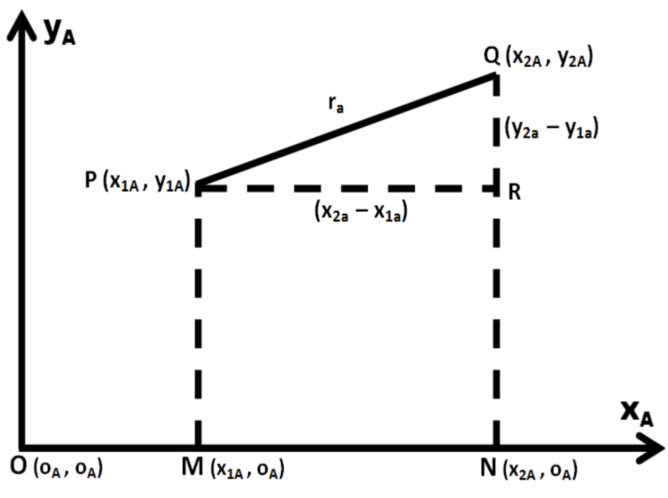

Figure 12. Distance between two object points

5.4 Equation of a line

General equation of an object line whose slope is m_a is

$$y_A = m_a \bullet x_A \oplus c_A .$$

Equation of an object line having slope m_a and passing through the object point $Q(x_{1A}, y_{1A})$ is $(y_A \sim y_{1A}) = m_a \bullet (x_A \sim x_{1A})$.

Equation of an object line passing through the two object points $P(x_{1A}, y_{1A})$ and $Q(x_{2A}, y_{2A})$ is $(y_A \sim y_{1A}) = m_a \bullet (x_A \sim x_{1A})$, where $m_a = (y_{2a} - y_{1a}) /(x_{2a} - x_{1a})$.

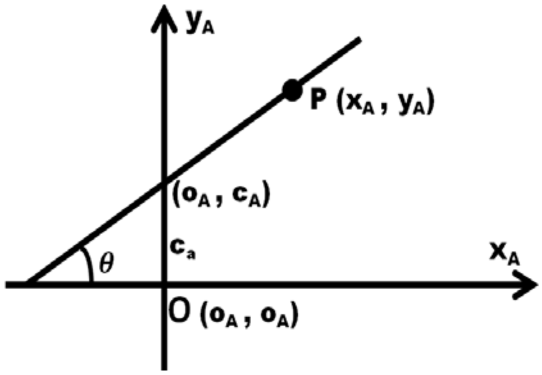

Figure 13. An object line having intercept of length c_a on Y_A axis

5.5 Object circle on the object plane

Equation of an object circle with center at $(0_A, 0_A)$ and radius r_a is

$$x_a^2 + y_a^2 = r_a^2$$

$$x^2 .1_a^2 + y^2 .\, 1_a^2 = r^2 .1_a^2$$

or

$$x^2 + y^2 = r^2$$

(in the region A).

Thus $x^2 + y^2 = r^2$ represents the equation of the object circle C_1 with radius r_a and center at $(0_A, 0_A)$ on the object plane of region A. Again, $x^2 + y^2 = r^2$ does also represent the equation of the object circle C_2 with radius r_b and center at $(0_B, 0_B)$ on the object plane of region B, and similarly $x^2 + y^2 = r^2$ is also the equation of the object circle C_3 with radius r_{rr} $(= r)$ and center at $(0_{RR}, 0_{RR})$, i.e., at $(0, 0)$ on the object plane of region RR, etc. for different complete regions. Each of these distinct circles C_1, C_2 and C_3 has the equation $x^2 + y^2 = r^2$ but of different radii as they are on different object planes. However, the circle C_3 is our classical circle of classical plane geometry.

Thus the equation $x^2 + y^2 = r^2$ of circle is region-dependent. If one asks the questions: "What is the radius and center of the circle $x^2 + y^2 = r^2$? What is the area of it?", then we cannot answer immediately unless we know the identity of the concerned object plane. Consequently, these questions are incomplete questions.

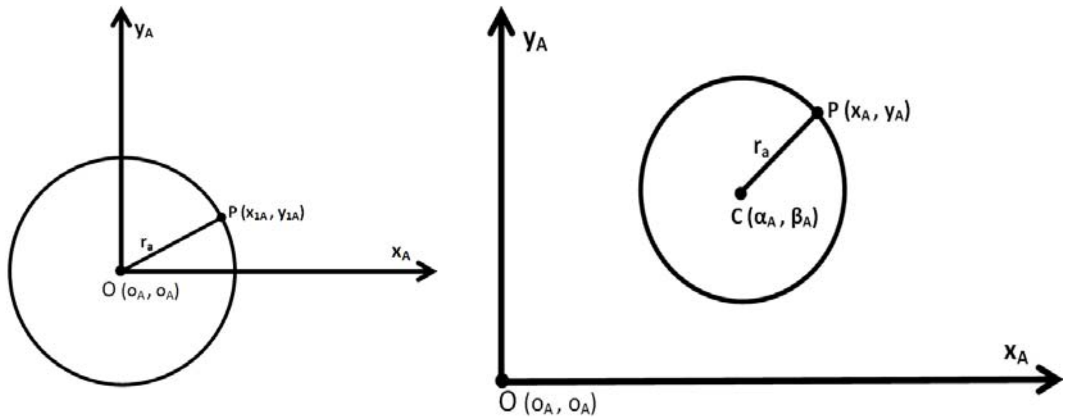

Figure 14(a), (b). Objects circles

If $1_a > 1$, then the object circle $x^2 + y^2 = r^2$ in the object plane of region A is a bigger circle than the classical circle $x^2 + y^2 = r^2$; if $1_a < 1$, then the object circle $x^2 + y^2 = r^2$ in the object plane of region A is a smaller circle than the classical circle $x^2 + y^2 = r^2$; and if $1_a = 1$, then the object circle $x^2 + y^2 = r^2$ is of same size with the classical circle $x^2 + y^2 = r^2$.

Equation of an object circle with center at (α_A, β_A) and radius r_a is:

$$(y_a - \beta_a)^2 + (x_a - \alpha_a)^2 = r_a^{\,2}$$

$$\text{or, } (y - \beta)^2 + (x - \alpha)^2 = r^2$$

Thus $(y - \beta)^2 + (x - \alpha)^2 = r^2$ represents the equation of the object circle C_1 with radius r_a and center at (α_A, β_A) on the object plane of region A. Again, $(y - \beta)^2 + (x - \alpha)^2 = r^2$ does also represent the equation of the object circle C_2 with radius r_b and center at (α_B, β_B) on the object plane of region B, and similarly $(y - \beta)^2 + (x - \alpha)^2 = r^2$ is also the equation of the object circle C_3 with radius r_{rr} $(= r)$ and center at $(\alpha_{RR}, \beta_{RR})$ i.e. at (α, β) on the object plane of region RR, etc. for different complete regions. Each of these distinct circles C_1, C_2 and C_3 has the equation $(y - \beta)^2 + (x - \alpha)^2 = r^2$ but of different radii as they are on different object planes. However, the circle C_3 is our classical circle of classical plane geometry. Thus the equation

$$(y - \beta)^2 + (x - \alpha)^2 = r^2$$

of circle is region-dependent.

The classical geometry (2-D geometry, 3-D or higher dimensional geometry) being practiced by the world mathematicians at elementary (see [14]) or higher level is a particular case of the ‘Object Geometry’.

6 Conclusion

The classical “Theory of Numbers” and “Geometry” developed so far is mainly based on the set R of real numbers, extended with two infinities (and then took their expanding shape time to time with the growth of advanced higher mathematics). The growth of these two giant subjects at every stage so far required fluent applications of various operations and results which are valid in the set of real numbers. In the existing literatures of the classical "Theory of Numbers" and "Geometry" of the subject Mathematics, R is assumed to be just a field or division algebra. But this assumption is not appropriate (rather, say 'not sufficient') because of the fact that using the properties of a 'field' or a 'division algebra' or of any existing algebra, many of the formulas, rules, results or materials of elementary as well as higher algebra cannot have the validity as shown with several examples in [2]. A careful study of the region algebra [2] will clarify that many of the results, formulas, equalities, identities, rules, etc. of elementary algebra (say, the algebra practiced at high school level or higher level) are not valid in the fields or in division algebras or in any existing algebras in general, but in regions only. It is observed that a region is a **"Commutative Division F-Algebra"**, but defined independently and uniquely with a self-identity here, with its important properties and results.

Fortunately the set R happens to be a trivial example of real region called by region RR [2]. And in a hidden way R has been providing the world mathematicians all the properties of region algebra, not just the limited properties of division algebra or of any of the existing algebra. Interestingly, the field R (or, the division algebra R) satisfies few additional properties trivially, but not by virtue of the properties of division algebra, by which it qualifies to become a real region. And consequently the development and continuous growth of classical Geometry or the classical Calculus never faced any computational constraints or invalidity, even assuming R to be a division algebra or any of the existing algebra just. The region algebra is applicable in "NR-Statistics" introduced in [4] to define and study various region measures like: region mean, region standard deviation, region variance, etc. with algebraic approach (in the Algebraic Statistics part of NR-Statistics). The region algebra is also applied in Data Structure for Big Data [7, 8], in the Theory of Solid Matrices and Solid Latrices [8].

In this work we have made further study of the "Theory of Objects" by introducing two new theories called by "Theory of A-numbers" and "Object Geometry". These two topics will surely grow in a lot of volume with time, the present work being just an initialization. We have identified 'What are the minimum properties which need to be satisfied by a set A so that a new Geometry can be developed over the platform A?' It has been explained how the classical "Theory of Numbers" being practiced by the world so far happens to be a topic of "Theory of RR-numbers" where the "Theory of RR-numbers" is a particular instance of our new "Theory of A-Numbers" of a complete region A. For a non-example, the set of all triangular fuzzy numbers [5] do not form a real region with respect to its commonly used operations, and hence cannot open any platform to develop any calculus as mentioned in [3], cannot open any new Theory of Numbers or a new Geometry at the present form. The set of all triangular fuzzy numbers (or the set of all trapezoidal fuzzy numbers)) is closed with respect to the addition operation defined over them, but is not closed with respect to the multiplication operation defined over them [5].

But initially, for the sake of presentation and to avoid any confusion we start with the following three theories sequentially. Each of these three theories are complete within itself.

Theory-1 (Theory of Numbers): The existing "Theory of Numbers" in the literature (on real numbers and complex numbers).

Theory-2 (Theory of Objects): Combinatorics on Region Algebra.

Theory of A-numbers: It is a part of Theory-2. Corresponding to every complete region A, there is a unique Theory of Numbers which is called by Theory of A-numbers. The 'Theory of Numbers' of one complete region is different from the 'Theory of Numbers' of another complete region. Thus the "Theory of A-numbers" and the "Theory of B-numbers" corresponding to two different complete regions A and B respectively are two independent theories. But all such theories fall under the Theory of Objects.

Consider a particular region: the complete region RR. Corresponding to this complete region RR, there is a unique Theory of Numbers which is called by "Theory of RR-numbers". This is designated her by **Theory-3** as mentioned below.

Theory-3 (Theory of RR-numbers): Theory-3 is all about the "Theory of RR-numbers", which is a particular case of the "Theory of A-numbers" in Theory-2 where the region A is RR here. In fact, Theory-1 happens to be a special case of Theory-3, but initiating of Theory-3 done by the author is not with the purpose of 'Making a generalization of the Theory-1'. It may also be noted that the Theory-3 is a special case of "Theory of A-numbers" where "Theory of A-numbers" is derived from the "Theory of Objects".

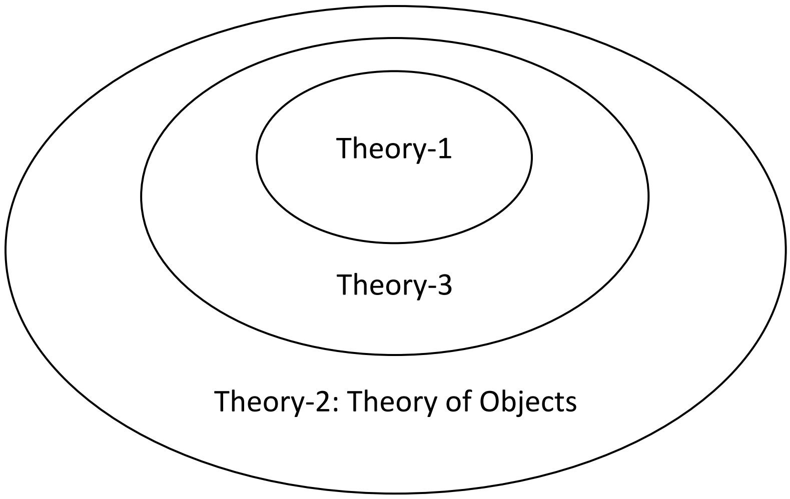

Figure 15. The three theories in the Venn diagram

Finally, we have arrived at a unified unique theory called by "**Theory of Objects"** at the end.

Acknowledgement

The author is indebted to all the Referees for their valuable comment to modify and improve the first version of this paper.

References

[1] Artin, M. (1991) *Algebra*, Prentice Hall, New York.

[2] Biswas, R. (2012) Region Algebra, Theory of Objects & Theory of Numbers, *International Journal of Algebra*, 6(28), 1371–1417.

[3] Biswas, R. (2013) Calculus Space, *International Journal of Algebra*, 7(16), 791–801.

[4] Biswas, R. (2014) Introducing Soft Statistical Measures, *The Journal of Fuzzy Mathematics,* 22(4).

[5] Biswas, R. (2012) Fuzzy Numbers Redefined, *Information*, 15(4), 1369–1380.

[6] Biswas, R. (2014) Birth of Compound Numbers, *Turkish Journal of Analysis and Number Theory*, 2(6), 208–219.

[7] Biswas, R. (2012) Heterogeneous Data Structure "r-Atrain", *INFORMATION: An International Journal (Japan)*, 15(2), 879–902.

[8] Biswas, R. (2013) Theory of Solid Matrices & Solid Latrices, Introducing New Data Structures MA, MT: for Big Data, *International Journal of Algebra*, 7(16), 767–789.

[9] Copson, E. T. (1968) *Metric Spaces*, Cambridge University Press.

[10] Dixon, G. M. (2010) *Division Algebras: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics*, Kluwer Academic Publishers, Dordrecht.

[11] Herstein, I. N. (2001) *Topics in Algebra*, Wiley Eastern Limited, New Delhi.

[12] Jacobson, N. (1985) *Basic Algebra I*, 2nd Ed., W. H. Freeman & Company Publishers, San Francisco.

[13] Jacobson, N. (1989) *Basic Algebra II*, 2nd Ed., W. H. Freeman & Company Publishers, San Francisco.

[14] Loney, S.L. (1975) *The Elements of Coordinate Geometry Part-I*, Macmillan Student Edition, Macmillan India Limited, Madras.

[15] Rudin, W. (2006) *Real and Complex Analysis*, McGraw Hills Education, India.

[16] Simmons, G.F. (1963) *Introduction to Topology and Modern Analysis*, McGraw Hill, New York.

[17] Van der Waerden, B. L. (1991) *Algebra*, Springer-Verlag, New York.