On $\Pi_k$–connectivity of some product graphs

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Abstract: Let $k$ be a positive integer. A graph $G = (V, E)$ is said to be $\Pi_k$-connected if for any given subset $S$ of $V(G)$ with $|S| = k$, the subgraph induced by $S$ is connected. In this paper, we consider $\Pi_k$-connected graphs under different graph valued functions. $\Pi_k$-connectivity of Cartesian product, normal product, join and corona of two graphs have been obtained in this paper.

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1 Introduction

Unless mentioned otherwise, for terminology and notation the reader may refer to Harary [3], new ones will be introduced as and when found necessary.

In this article, we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set $V$ and edge set $E$. As such $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph $G$, respectively. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices $X \subseteq V$. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex $v$, respectively. A non-trivial graph $G$ is called connected if any two of its vertices are linked by a path in $G$.

A $\Pi_k$-connected graph $G$ is said to be vertex minimal $\Pi_k$-connected if $G$ is not $\Pi_{k-1}$-connected. A vertex minimal $\Pi_k$-connected graph $G$ is said to be partially vertex–edge minimal $\Pi_k$-connected
if \( G - e \) is not \( \Pi_k \)-connected for atmost \(|E(G)| - 1 \) edges of \( G \). A vertex minimal \( \Pi_k \)-connected graph \( G \) is said to be totally vertex–edge minimal \( \Pi_k \)-connected if \( G - e \) is not \( \Pi_k \)-connected for every \( e \in E(G) \).

The Cartesian product of two graphs \( G \) and \( H \), denoted \( G \square H \), is a graph with vertex set \( V(G \square H) = V(G) \times V(H) \), that is, the set \( \{(g, h) / g \in G, h \in H\} \).

The edge set of \( G \square H \) consists of all pairs \([ (g_1, h_1), (g_2, h_2) ] \) of vertices with \([ g_1, g_2 ] \in E(G) \) and \( h_1 = h_2 \), or \( g_1 = g_2 \) and \([ h_1, h_2 ] \in E(H) \).

The normal product of two graphs \( G \) and \( H \), denoted \( G \bigoplus H \), is a graph with vertex set \( V(G \bigoplus H) = V(G) \times V(H) \), that is, the set \( \{(g, h) / g \in G, h \in H\} \), and an edge \([ (g_1, h_1), (g_2, h_2) ] \) exists whenever any of the following conditions hold good:
(i) \([ g_1, g_2 ] \in E(G) \) and \( h_1 = h_2 \),
(ii) \( g_1 = g_2 \) and \([ h_1, h_2 ] \in E(H) \),
(iii) \([ g_1, g_2 ] \in E(G) \) and \([ h_1, h_2 ] \in E(H) \). Given a digraph \( G_0 = (V_0, E_0) \) and a family of digraphs \( \{G_v = (V_v, E_v)\}_{v \in V_0} \) indexed by \( V_0 \), the generalized lexicographic product, denoted by \( G_0 \bigoplus \{G_v \}_{v \in V_0} \) is defined as the digraph with vertex set \( V = \{(u, v) / u \in V_0 \text{ and } v \in V_v \} \) and arc set \( E = \{((u, v), (v', w')) / (v, v') \in E_0 \text{ or } (v = v') \text{ and } (w, w') \in E_v \} \).

Join of two graphs is denoted by \( G_1 + G_2 \) and consists of \( G_1 \cup G_2 \) and all edges joining \( V_1 \) with \( V_2 \). The corona \( G_1 \circ G_2 \) was defined by Frucht and Harary [1] as the graph \( G \) obtained by taking one copy of \( G_1 \) of order \( p_1 \) and \( p_1 \) copies of \( G_2 \), and then joining the \( i' \)th node of \( G_1 \) to every node in the \( i' \)th copy of \( G_2 \).

### 2 Preliminary results

**Theorem 2.1.** A connected graph \( G \) is vertex minimal \( \Pi_{p'} \)-connected if and only if it has at least one cut vertex.

### 3 Main results

**Proposition 3.1.** Any graph \( G \) is \( \Pi_k \)-connected if and only if every subgraph of \( G \) having order at least \( k \) is \( \Pi_k \)-connected.

**Proof.** Let \( G \) be any \( \Pi_k \)-connected graph. On contrary, suppose there exists a disconnected vertex induced subgraph \( H \) of order at least \( k \). Form a \( k \) - vertex subset \( S \) of \( V(H) \) by taking at least one vertex from at least two components of \( H \). The subgraph induced by \( S \) is disconnected, a contradiction.

Conversely, suppose every subgraph of order at least \( k \) is \( \Pi_k \)-connected. Hence, every subgraph of order \( k \) is connected. Therefore, \( G \) is \( \Pi_k \)-connected. \( \square \)

**Theorem 3.1.** Prism of a complete graph \( K_p \) is totally vertex–edge minimal \( \Pi_{p+1} \)-connected.

**Proof.** Let \( G \) be the prism of a complete graph \( K_p \). Let \( G_1 \) and \( G_2 \) be the copies of \( K_p \) in the prism. Let \( T \) be an arbitrary set of \( p + 1 \) vertices, hence \( T \) contains at least two adjacent vertices.
$u, v$ such that $u \in V(G_1)$ and $v \in V(G_2)$, where $\langle V(G_1) \rangle$ and $\langle V(G_2) \rangle$ are complete graphs. Hence, the subgraph induced by $T$ is connected. Now we prove $G$ is not $\Pi_p$-connected and $G - e$ is not $\Pi_{p+1}$-connected. Now we shall prove $G$ is not $\Pi_p$-connected. Let $M$ be the set of vertices consisting of $p - 1$ vertices from $V(G_1)$ and a vertex from $V(G_2)$ non-adjacent with any of the $p - 1$ vertices. The subgraph induced by $M$ is disconnected and hence $G$ is not $\Pi_p$-connected. Now we prove $G - e$ is not $\Pi_{p+1}$-connected. Here two cases arise:

- **Case(1):** $e \in G_1$ or $e \in G_2$, and
- **Case(2):** $e$ is an edge between $G_1$ and $G_2$.

**Case(1):** Let $S \subset V(G_2)$ consist of $p - 1$ vertices. Let $u, v \in V(G_1)$ be such that $u$ is not adjacent to any of the vertices of $S$ and $v \in V(G_1)$ be any vertex. In $G - e(=uv)$, the subgraph induced by $\langle S \cup \{u, v\} \rangle$ is disconnected.

**Case(2):** Let $e = uf(u)$ be any edge between $G_1$ and $G_2$. In $G - e(=uf(u))$, the subgraph induced by $V(G_2) \cup u$ is not connected and hence $G - e(=uf(u))$ is not $\Pi_{p+1}$-connected. Hence, $G$ is totally vertex–edge minimal $\Pi_{p+1}$-connected.

**Remark 1.** Every totally vertex–edge minimal graph is partially vertex–edge minimal but every partially vertex–edge minimal need not be totally vertex–edge minimal.

**Theorem 3.2.** Prism of partially vertex–edge minimal $\Pi_3$-connected graph of order $p \geq 3$ is partially vertex–edge minimal $\Pi_{p+2}$-connected.

**Proof.** Let $G$ be a partially vertex–edge minimal $\Pi_3$-connected graph of order $p$ and $H$ be the prism of $G$. Let $G_1$ and $G_2$ be two copies of $G$ in the prism of $G$. Let the matching be the union of edges $(u, f(u))$ for all $u$ in $G_1$ and $f(u)$ in $G_2$, where $f : V(G_1) \rightarrow V(G_2)$ is a bijection from $V(G_1)$ to $V(G_2)$ such that $f(u)$ is the mirror image of $u$. Let $T \subset V(H)$ be any subset having $p + 2$ vertices. It is clear that $T$ contains at least four vertices $u, v, f(u)$ and $f(v)$ such that $u$ is adjacent to $f(u)$ and $v$ is adjacent to $f(v)$. Here two cases arise:

- **Case(1):** $u$ is adjacent to $v$, and
- **Case(2):** $u$ is not adjacent to $v$.

**Case(1):** As $G_1$ is $\Pi_3$-connected, every vertex in $T \cap V(G_1)$ is adjacent to at least one of the two vertices $u, v$. Similarly, every vertex in $T \cap V(G_2)$ is adjacent to at least one of the two vertices $f(u), f(v)$. Hence, the subgraph induced by the set $T$ is connected.

**Case(2):** Every vertex in $T \cap V(G_1)$ is a common neighbor of $u$ and $v$ and every vertex in $T \cap V(G_2)$ is a common neighbor of $f(u)$ and $f(v)$. Therefore the subgraph induced by the set $T$ is connected. Hence, the prism of $G$ is $\Pi_{p+2}$ connected. $H$ is not $\Pi_{p+1}$ connected, as the graph induced by $\{V(G_1) - u\} \cup \{f(u), f(v)\}$ is disconnected, where $f(u)$ is not adjacent to $f(v)$. Since $G_1$ is partially vertex–edge minimal $\Pi_3$-connected, there exists an edge $e \in E(G_1)$ such that $G_1 - e$ is not $\Pi_3$-connected, i.e., there exists three vertices $u, v$ and $w$ in $V(G_1 - e)$ such that the subgraph induced by $u, v$ and $w$ is disconnected. Let $u$ be a vertex not adjacent to $v$ and $w$. 

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Hence, the subgraph of \( H - e \) induced by \( \{V(G_2) - f(u)\} \cup \{u, v, w\} \) is disconnected. Hence, \( H - e \) is not \( \Pi_{p+2} \)-connected.

**Case (a):** Suppose \( e \in V(G_1) \). Since \( G_1 \) is partially vertex–edge minimal \( \Pi_3 \)-connected graph, \( G_1 - e \) is not \( \Pi_3 \)-connected, i.e., there exist three vertices \( u, v, \) and \( w \) whose induced graph is not connected. The subgraph induced by subset \( \{V(G_2) - f(u)\} \cup \{u, v, w\} \) of \( V(H - e) \) is not connected. Hence, \( H - e \) is not \( \Pi_{p+2} \)-connected. Similarly we can prove \( H - e \) is not \( \Pi_{p+2} \)-connected when \( e \in V(G_2) \).

**Case (b):** Suppose \( e = (u, f(u)) \) is contained in the matching. Since \( G_2 \) is partially vertex–edge minimal \( \Pi_3 \)-connected graph, there exist a vertex \( f(v) \) non-adjacent to \( f(u) \). The subgraph of \( H - e \) induced by \( \{f(u), f(v)\} \) is not connected. Hence, \( H - e \) is not \( \Pi_{p+2} \)-connected. Also \( H \) is not \( \Pi_{p+1} \) connected, as the subgraph induced by \( \{V(G_1) - u\} \cup \{f(u), f(v)\} \), where \( f(u) \) is not adjacent to \( f(v) \), is disconnected.

Example:

![Prism of partially vertex–edge minimal \( \Pi_3 \)-connected graph having even order](image)

**Figure 1:** Prism of partially vertex–edge minimal \( \Pi_3 \)-connected graph having even order

**Theorem 3.3.** Prism of totally vertex–edge minimal \( \Pi_3 \)-connected graph having even order is totally vertex–edge minimal \( \Pi_{p+2} \) connected.

**Proof.** Let \( G \) be the prism of totally vertex–edge minimal \( \Pi_3 \)-connected graph having even order \( p \). \( G \) is:

(i) \( \Pi_{p+2} \) connected but not \( \Pi_{p+1} \) connected;
(ii) \( G - e_{G_1}, G - e_{G_2} \) and \( G - e_{G_1,G_2} \) are not \( \Pi_{p+2} \).

Hence, the prism is totally vertex–edge minimal \( \Pi_{p+2} \) connected.

\( \square \)

**Note:** In every totally vertex–edge minimal \( \Pi_3 \)-connected graph having odd order \( p \), there exist exactly one vertex having degree \( p - 1 \).

**Theorem 3.4.** Prism of totally vertex–edge minimal \( \Pi_3 \)-connected graph having odd order \( p \) is partially vertex–edge minimal \( \Pi_{p+2} \) connected.

**Proof.** Let \( H \) be a totally vertex–edge minimal \( \Pi_3 \)-connected graph having odd order \( p \) and \( v \) be a vertex in \( H \) having degree \( p - 1 \). Let \( G \) be the prism of \( H \). \( G \) is:

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(i) $\Pi_{p+2}$ connected but not $\Pi_{p+1}$ connected;
(ii) $G - e_{G_1}$ is not $\Pi_{p+2}$ connected, $e$ is any edge in $G_1$ not incident with a vertex $v$, but $G - e_{G_1}$ is still $\Pi_{p+2}$ connected, if $e$ is incident with $v$.

Hence, $G$ is partially vertex–edge minimal $\Pi_{p+2}$ connected.

The following construction gives the class of non-regular $\Pi_k$-connected graphs, where $k$ is a function of $p$.

**Theorem 3.5.** There exist a non-regular $\Pi_{2n+3}$-connected graph on $2(2n) + 1$, $n \geq 3$ vertices.

*Proof.* Let $G$ be a totally vertex–edge minimal $\Pi_3$-connected graph of order $2n$ and $G'$ be a prism of $G$. Let $G_1$ be the graph obtained by adding a vertex $v$ and making it adjacent to every vertex of one of the copies in the prism. The resulting graph $G_1$ is non-regular since $deg(v) = 2n$ and degree of other vertices is $2n - 1$, as $v$ is adjacent to all the vertices in the second copy of the prism. $G_1 - v$ is isomorphic to the graph obtained in theorem(2.3), which is totally vertex–edge minimal $\Pi_{p+2}$ connected. Let $T$ be any set of $2n + 3$ vertices in $G_1$. Here two cases arise:

- **Case (i):** $v$ belongs to $T$, and
- **Case (ii):** $v$ does not belong to $T$.

**Case (i):** Suppose $v$ belongs to $T$. The subgraph induced by $T - v$ is connected and contains at least four vertices $u, w$ from the first copy and $f(u), f(w)$ from the second copy such that $u$ is adjacent to $f(u)$ and $w$ is adjacent to $f(w)$ and hence $\langle T \rangle$ is connected as $v$ is adjacent to all the vertices in the second copy.

**Case (ii):** Suppose $v$ does not belong to $T$. Clearly the subgraph $\langle T \rangle$ is connected as every $\Pi_k$ graph is $\Pi_{k+1}$ also. Hence, $G_1$ is $\Pi_{2n+3}$-connected graph. 

Example:

![Figure 2: Prism of $\Pi_3$-connected graph having even order](image)

**Theorem 3.6.** Cartesian product $K_{p_1} \times K_{p_2}$ of two complete graphs $K_{p_1}$ and $K_{p_2}$ is totally vertex–edge minimal $\Pi_k$-connected, where $k = p_1p_2 - p_1 - p_2 + 3$. 

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Proof. Let $V_1$ be the set of vertices in $K_{p_1}$ and $V_2$ be the set of vertices in $K_{p_2}$. Let $u \in V_1$ and $v \in V_2$. The subgraph induced by the vertices $\{V_1 - u\} \times \{V_2 - v\} \cup (u, v)$ is a disconnected graph. If we include one more vertex, say $(u, w)$ in the above set, the subgraph becomes connected. Hence, the Cartesian product $K_{p_1} \times K_{p_2}$ is vertex minimal $\Pi_k$-connected, where $k = p_1p_2 - p_1 - p_2 + 3$ and if we remove $e = (u, w)$, the above subgraph becomes disconnected. Since the vertices and the edge are arbitrarily chosen, the Cartesian product $K_{p_1} \times K_{p_2}$ is totally vertex–edge minimal $\Pi_k$-connected, where $k = p_1p_2 - p_1 - p_2 + 3$. Hence, the proof.

Remark 2. In the normal product $K_p \bigoplus G$ of a complete graph and any graph $G$, the subgraph induced by the set of vertices $A \cup B$, where $A = \{(u, x)/ u \in V(K_p)\}$ and $B = \{(u, y)/ u \in V(K_p)\}$ is complete bipartite whenever $x$ is adjacent to $y$ in $G$.

Remark 3. Cartesian product of two connected graphs is connected if and only if both are connected.

Theorem 3.7. If the graph $G$ is vertex minimal $\Pi_k$-connected, $k \geq 3$ then the normal product $K_p \bigoplus G$ is $\Pi_{p(k-1)+1}$-connected.

Proof. Let a graph $G$ be $\Pi_k$-connected. Since $G$ is vertex minimal $\Pi_k$-connected, there exist a set $(S)$ of $k - 1$ vertices whose induced subgraph is disconnected and hence the subgraph of $K_p \bigoplus G$ induced by $\langle V(K_p) \times S \rangle$ is disconnected. Any vertex $w$ not in $S$ makes the subgraph induced by $(S \cup w)$ connected and hence from the above remark, the subgraph induced by $\langle V(K_p) \times S \cup (u, w) \rangle$ is connected, where $u$ is any vertex in $K_p$. Hence, the normal product $K_p \bigoplus G$ is $\Pi_{p(k-1)+1}$-connected.

Theorem 3.8. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are $\Pi_m$ and $\Pi_n$-connected graphs respectively, then $G_1 \bigoplus G_2$ is:

(i) $\Pi_{p_1(n-1)+1}$-connected, if $p_1(n-1) \geq p_2(m-1)$;
(ii) $\Pi_{p_2(m-1)+1}$-connected, if $p_2(m-1) > p_1(n-1)$.

Proof. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are $\Pi_m$ and $\Pi_n$-connected graphs respectively and $p_1(n-1) > p_2(m-1)$. Let $T = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$ be any set of $p_1(n-1) + 1$ vertices in $G_1 \bigoplus G_2$. Since $p_1(n-1) > p_2(m-1)$, there exists at least $m$ distinct $x_i$’s and at least $n$ distinct $y_i$’s in $T$. Suppose the subgraph induced by $T$ is disconnected, then the subgraphs induced by $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are disconnected in $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$, a contradiction. Hence, the subgraph induced by $T$ is connected.

Similarly we can prove the second case.
**Remark 4.** Let the graphs $G_1(p, q_1)$ and $G_2(p, q_2)$ be $\Pi_{k_1}$-connected with $|V(G_1)| < |V(G_2)|$ then $G_1 \bigoplus G_2$ is $\Pi_{|V(G_2)|(|k_1|-1)+1}$-connected.

**Theorem 3.9.** Let the graphs $G_1$ and $G_2$ are $\Pi_{k_1}$ and $\Pi_{k_2}$-connected with $|V(G_1)| \leq |V(G_2)|$ and $k_1 > k_2$ then $G_1 \bigoplus G_2$ is $\Pi_{|V(G_2)|(|k_1|-1)+1}$-connected.

**Proof.** Let the graphs $G_1$ and $G_2$ are $\Pi_{k_1}$ and $\Pi_{k_2}$-connected with $|V(G_1)| \leq |V(G_2)|$ and $k_1 > k_2$. The normal product $G_1 \bigoplus G_2$ is not $\Pi_{|V(G_2)|(|k_1|-1)}$-connected since $G_1$ is $\Pi_{k_1}$-connected, there exist at least one subset $S$ containing $k_1 - 1$ such that the subgraph $\langle S \rangle$ induced by $S$ is disconnected and hence the subgraph $\langle S \times v(G_2) \rangle$ induced by $S \times v(G_2)$ is disconnected, implies the normal product $G_1 \bigoplus G_2$ is not $\Pi_{|V(G_2)|(|k_1|-1)}$-connected and $\langle S \times V(G_2) \cup \{u, v\} \rangle$ is connected, where $S$ and $(u, v)$ are arbitrarily chosen. Hence, $G_1 \bigoplus G_2$ is $\Pi_{|V(G_2)|(|k_1|-1)+1}$-connected. 

**Theorem 3.10.** Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be $\Pi_{k_1}$ and $\Pi_{k_2}$-connected graphs respectively, then the join of $G_1$ and $G_2$ is $\Pi_{k_3}$-connected, where $k_3 = \max\{k_1, k_2\}$.

**Proof.** Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be $\Pi_{k_1}$ and $\Pi_{k_2}$-connected graphs respectively. The join is not $\Pi_{k_3-1}$-connected because when all the $k_3 - 1$ vertices belongs to $V(G_2)$, where $k_3 = k_2$, then subgraph induced by these $k_3 - 1$ vertices is disconnected. Hence, the join is $\Pi_{k_3}$-connected.

**Theorem 3.11.** Let $G_0 = (V_0, E_0)$ be a graph and $\{G_v = (V_v, E_v)\}_{v \in V_0}$ indexed by $V_0$ be a family of isomorphic graphs then the generalized lexicographic product $G_0\{G_v\}_{v \in V_0}$ is $\Pi_t$-connected, where $t = (k - 1)|V_0| + 1$ if and only if $G_0$ is $\Pi_k$-connected, $k \geq 2$.

**Proof.** Let $G_0$ be a $\Pi_k$-connected graph. The lexicographic product $G_0\{G_v\}_{v \in V_0}$ is not $\Pi_{(k-1)|V_0|}$-connected as the graph $G_0$ is not $\Pi_{k-1}$-connected, there exist at least one set of $k - 1$ vertices in $V(G_0)$ whose induced subgraph is disconnected in $G_0$, replacing each of these $k - 1$ vertices by $G_v$, we get a disconnected $(k - 1)|V_0|_{\text{vertex induced subgraph}}$ of the generalized lexicographic product $G_0\{G_v\}_{v \in V_0}$. The set of $(k - 1)|V_0| + 1$ vertices are to be chosen from at least $k$ number of $G_v^*$ and hence the subgraph induced by any $(k - 1)|V_0| + 1$ vertices is connected as $G_0$ is $\Pi_k$-connected graph.
Conversely, let $G_0[\{G_v\}_{v \in V_0}]$ be $\Pi_t$-connected. Suppose on contradiction that $G_0$ is not $\Pi_k$-connected, then there exist at least one set, say $S$ whose cardinality is greater than $k$. The subgraph, say $G'$ induced after replacing each vertex of $S$ by the set of vertices $V_v$, is disconnected, i.e., there exists a set of $k$ vertices on which the subgraph induced is not connected and hence by replacing each vertex of $S$ by $V(G_v)$, we get a disconnected subgraph on $(k)|V_v|$, induced in $G_0[\{G_v\}_{v \in V_0}]$, a contradiction to our assumption $G_0[\{G_v\}_{v \in V_0}]$ is $\Pi_t$-connected. Hence, $G_0$ is $\Pi_k$-connected.

**Theorem 3.12.** Tensor product of two complete graphs $K_{p_1}$, $p_1 \geq 3$ and $K_{p_2}'$, $p_2' \geq 2$ is $\Pi_k$-connected, where $k = p_1 + p_2'$.

**Proof.** We first prove the tensor product of two complete graphs $K_{p_1}$ and $K_{p_2}$ is not $\Pi_{k-1}$-connected.

Suppose all the $k - 1$ vertices lie on the column and row. Then the vertex lying at the intersection of these column and row is not adjacent to any of the vertices lying in these two row and column and hence the subgraph induced by these $k - 1$ vertices is disconnected. Now we prove $k$ is a minimum such that the tensor product is $\Pi_k$ connected.

Let $S$ be any set of $k$ vertices in the tensor product. Take any vertex $v$ from $S$ and the corresponding row and column containing the vertex $v$. There exists at least one vertex $u$ not lying on these column and row and hence adjacent to $v$. Now here we take two cases:

- **Case (1):** There exists only one vertex $u$ not lying on these row and column.
- **Case (2):** There exists at least two vertices not lying on these row and column.

**Case (1):** Since the number of vertices in $S$ is $p_1 + p_2'$, there exists at least one vertex $x$ not lying on the row and column containing $u$. Again the following two sub-cases arise:

(i) $x$ lies on the row or column containing $v$ then $x$ will be adjacent to $u$ and $x$ is also adjacent to all the vertices of $S$ lying on column or row containing $v$. Hence, the subgraph in this case induced by $k$ vertices is connected.

(ii) $x$ not lying on row or column containing $v$. Then $x$ will be adjacent to $u$, also $x$ will be adjacent to at least one vertex in row or column containing $v$. Hence, in this case also the graph induced by $k$ vertices is connected.

**Case (2):** There exist at least two vertices not lying on these row and column. Suppose these vertices lie on same row or column then each will be adjacent to at least one vertex lying in column or row as the case may be and hence the subgraph induced by these $k$ vertices is connected. Now suppose these vertices lie on different rows or different columns then each vertex will be adjacent to the other vertices lying in other rows or columns. Hence, in this case also the subgraph induced by $k$ vertices is connected.
Theorem 3.13. For any two connected graphs \( G_1(p_1, q_1) \) and \( G_2(p_2, q_2) \), \( G_1 \circ G_2 \) is vertex minimal \( \Pi_{p_1p_2} \)-connected.

Proof. Let \( G_1(p_1, q_1) \) and \( G_2(p_2, q_2) \) be any two graphs. Now we prove the \( G_1 \circ G_2 \) is not \( \Pi_{p_1p_2-1} \). The subgraph induced by the vertices \( \{u_1, u_2, \ldots, u_{p_1-1}\} \cup \{V(G_{u_1}), V(G_{u_2}), \ldots, V(G_{u_{p_1}})\} \) is a disconnected subgraph of \( G_1 \circ G_2 \). Hence, \( G_1 \circ G_2 \) is vertex minimal \( \Pi_{p_1p_2} \)-connected.

References


