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On right circulant matrices with general number sequence

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Abstract: In this paper, the elements of the general number sequence were used as entries for right circulant matrices. The eigenvalues, the Euclidean norm and the inverse of the resulting matrices were obtained.
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1 Introduction

The general number sequence as defined in [1] is a sequence whose terms satisfy the recurrence relation

$$w_n = pw_{n-1} - qw_{n-2} \tag{1}$$

with initial values $w_0 = a$ and $w_1 = b$. Here, a, b, p and $q \in \mathbb{Z}$. The n-th term of the general number sequence is given by:

$$w_n = \frac{A\alpha^n + B\beta^n}{\alpha - \beta} \tag{2}$$

where

$$\begin{aligned} A &= b - a\beta \\ B &= a\alpha - b \\ \alpha + \beta &= p \\ \alpha\beta &= q \\ \alpha - \beta &= \sqrt{p^2 - 4q} \neq 0 \end{aligned}$$

The numbers α and β are the roots of the equation $x^2 - px + q = 0$.

A right circulant matrix with general number sequence is a matrix of the form

$$RCIRC_n(\vec{w}) = \begin{pmatrix} w_0 & w_1 & w_2 & \dots & w_{n-2} & w_{n-1} \\ w_{n-1} & w_0 & w_1 & \dots & w_{n-3} & w_{n-2} \\ w_{n-2} & w_{n-1} & w_0 & \dots & w_{n-4} & w_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_2 & w_3 & w_4 & \dots & w_0 & w_1 \\ w_1 & w_2 & w_3 & \dots & w_{n-1} & w_0 \end{pmatrix},$$

where w_k are the first n terms of the general number sequence.

2 Preliminary result

The following lemma will be used to prove one of the main results.

Lemma 2.1.

$$\sum_{k=0}^{n-1}(r\omega^{-m})^k = \frac{1-r^n}{1-r\omega^{-m}}$$

where $\omega = e^{2i\pi/n}$.

Proof: Note that $\sum_{k=0}^{n-1}(r\omega^{-m})^k$ is a geometric series with first term 1 and common ratio $r\omega^{-m}$. Using the formula for the sum of a geometric series, we have

$$\begin{aligned} \sum_{k=0}^{n-1}(r\omega^{-m})^k &= \frac{1-r^n\omega^{-mn}}{1-r\omega^{-m}} \\ &= \frac{1-r^n(\cos 2\pi + i\sin 2\pi)}{1-r\omega^{-m}} \\ &= \frac{1-r^n}{1-r\omega^{-m}} \end{aligned}$$

This completes the proof. □

3 Main results

Theorem 3.1. *The eigenvalues of* $RCIRC_n(\vec{w})$ *are given by*

$$\lambda_m = \frac{1}{\alpha-\beta}\left[\frac{A(1-\alpha^n)}{1-\alpha\omega^{-m}} + \frac{B(1-\beta^n)}{1-\beta\omega^{-m}}\right]$$

where $m = 0, 1, ..., n-1$.

Proof: From [2], the eigenvalues of a right circulant matrix are given by the Discrete Fourier transform

$$\lambda_m = \sum_{k=0}^{n-1} c_k\omega^{-mk} \tag{3}$$

where c_k are the entries in the first row of the right circulant matrix. Using this formula, the eigenvalues of $RCIRC_n(\vec{w})$ are

$$\begin{aligned}\lambda_m &= \sum_{k=0}^{n-1}\left[\frac{A\alpha^k + B\beta^k}{\alpha-\beta}\right]\omega^{-mk} \\ &= \frac{1}{\alpha-\beta}\sum_{k=0}^{n-1}\left[A\alpha^k + B\beta^k\right]\omega^{-mk}.\end{aligned}$$

Using Lemma 2.1 we get the desired equation. □

Theorem 3.2. *The Eucliden norm of $RCIRC_n(\vec{w})$ is given by*

$$\|RCIRC_n(\vec{w})\|_E = \frac{1}{|\alpha-\beta|}\sqrt{n\left[\frac{2AB(1-q^n)}{1-q} + \frac{A^2(1-\alpha^{2n})}{1-\alpha^2} + \frac{B^2(1-\beta^{2n})}{1-\beta^2}\right]}.$$

Proof:

$$\begin{aligned}\|RCIRC_n(\vec{w})\|_E &= \sqrt{n\sum_{k=0}^{n-1}\left[\frac{A\alpha^k + B\beta^k}{\alpha-\beta}\right]^2} \\ &= \sqrt{n\sum_{k=0}^{n-1}\left[\frac{A^2\alpha^{2k} + B^2\beta^{2k} + 2AB\alpha\beta}{(\alpha-\beta)^2}\right]} \\ &= \frac{1}{|\alpha-\beta|}\sqrt{n\sum_{k=0}^{n-1}[A^2\alpha^{2k} + B^2\beta^{2k} + 2AB\alpha\beta]} \\ &= \frac{1}{|\alpha-\beta|}\sqrt{n\sum_{k=0}^{n-1}[A^2\alpha^{2k} + B^2\beta^{2k} + 2ABq]}.\end{aligned}$$

Note that each term in the summation is from a geometric sequence, so using the formula for sum of geometric sequence, the theorem follows. □

Theorem 3.3. *The inverse of $RCIRC_n(\vec{w})$ is given by*

$$RCIRC_n\left(s_0,\ s_1,\ \ldots, s_{n-1}\right)$$

where

$$s_k = \frac{\alpha-\beta}{n}\sum_{m=0}^{n-1}\left[\frac{(1-\alpha\omega^{-m})((1-\beta\omega^{-m})\omega^{mk}}{A(1-\alpha^n)(1-\beta\omega^{-m}) + B(1-\beta^n)(1-\alpha\omega^{-m})}\right].$$

Proof: The entries of the inverse of a right circulant matrix can be solved using the Inverse Discrete Fourier transform

$$s_k = \frac{1}{n}\sum_{m=0}^{n-1}\lambda_m^{-1}\omega^{mk} \tag{4}$$

where λ_m are the eigenvalues of the right circulant matrix. Using this equation and Theorem 3.1 we have

$$\begin{aligned} s_k &= \frac{1}{n}\sum_{m=0}^{n-1}\left[\frac{1}{\alpha-\beta}\left[\frac{A(1-\alpha^n)}{1-\alpha\omega^{-m}}+\frac{B(1-\beta^n)}{1-\beta\omega^{-m}}\right]\right]^{-1}\omega^{mk} \\ &= \frac{\alpha-\beta}{n}\sum_{m=0}^{n-1}\left[\frac{A(1-\alpha^n)}{1-\alpha\omega^{-m}}+\frac{B(1-\beta^n)}{1-\beta\omega^{-m}}\right]^{-1}\omega^{mk} \\ &= \frac{\alpha-\beta}{n}\sum_{m=0}^{n-1}\left[\frac{(1-\alpha\omega^{-m})(1-\beta\omega^{-m})\omega^{mk}}{A(1-\alpha^n)(1-\beta\omega^{-m})+B(1-\beta^n)(1-\alpha\omega^{-m})}\right]. \end{aligned}$$

This completes the proof. □

References

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