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On some Pascal's like triangles. Part 9

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Abstract: In a series of papers, Pascal's like triangles with different forms have been described. Here, three-dimensional analogues of these triangles are given and some of their properties are studied.
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1 Introduction

Following the series of six papers [1–3, 5–7], in which we discussed a new type of Pascal's Like Triangles (PLTs), in [4 ,8] we have shown their 3-dimensional analogues in the form of 3- or 4-face pyramids. Now, we generalize the form of the 4-face pyramids.

Following [8], let us denote the three important lines of the (infinite, 4-face) pyramid as follows (see the figure below):

- A-lines – lines that correspond to the four (infinite) pyramid edges,
- B-lines – lines that correspond to the four (infinite) "bisector" faces,
- C-line – the line that corresponds to the (infinite) pyramid's "altitude".

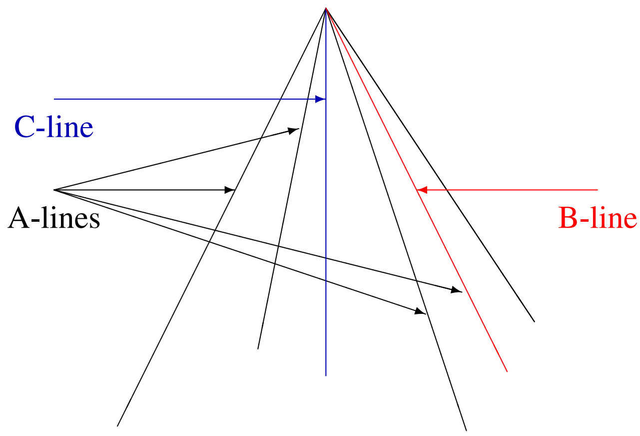

Figure 1.

In the present research, we discuss two versions of 4-face pyramids, that have the form given in Fig. 1, where there are nine lines:

- four A-lines – A_1-, A_2-, A_3- and A_4-lines,
- four B-lines – B_1-, B_2-, B_3- and B_4-lines, and
- one C-line.

Fig. 2 is further detailization of Fig. 1. In it, the points that correspond to the places of the sequences elements are shown. Now, we can mention that A-lines coincide with the left and right generating sequences and the B-lines – with the generating sequence in the standard PLT.

On the other hand, C-line corresponds to a generated sequence in a non-standard PTL. The way for obtaining the values of the members of this sequence will be discussed below.

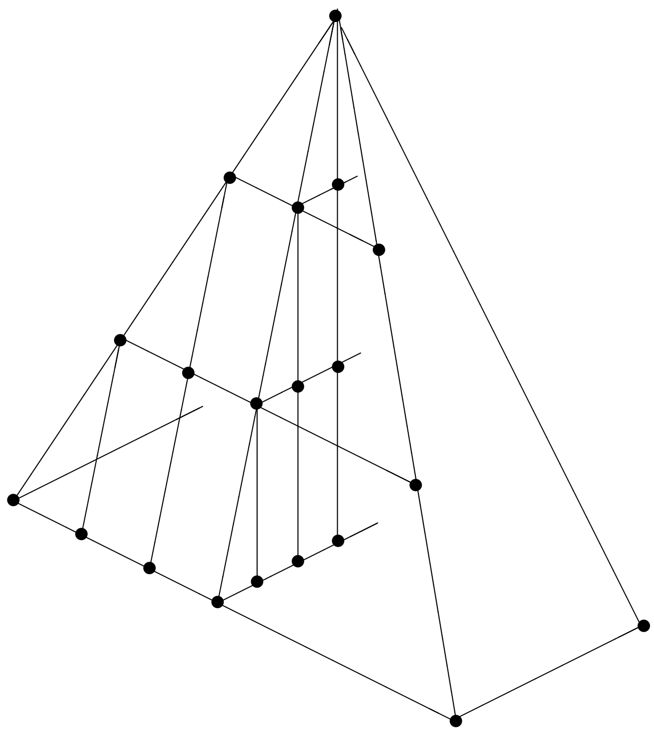

Figure 2.

In Fig. 3, we show the first two sequential horisontal sections (levels) of the pyramid, with the points in them, so that each point is symbolized appropriately, keeping, as far as possible, the numeration of the vertices from the previous papers.

Each of the A-, B- and C-lines, that, of course, here represent numerical sequences, has the same first element a. Let the elements of the seven sequences are written in the columns under the line-indices in the following Table.

$N.$	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C
1	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$
2	$\alpha_{2,1,1}$	$\alpha_{2,1,3}$	$\alpha_{2,3,1}$	$\alpha_{2,3,3}$	$\alpha_{2,1,2}$	$\alpha_{2,2,1}$	$\alpha_{2,2,3}$	$\alpha_{2,3,2}$	$\alpha_{2,2,2}$
3	$\alpha_{3,1,1}$	$\alpha_{3,1,5}$	$\alpha_{3,5,1}$	$\alpha_{3,5,5}$	$\alpha_{3,1,3}$	$\alpha_{3,3,1}$	$\alpha_{3,3,5}$	$\alpha_{3,5,3}$	$\alpha_{3,3,3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\alpha_{n,1,1}$	$\alpha_{n,1,2n-1}$	$\alpha_{n,2n-1,1}$	$\alpha_{n,2n-1,2n-1}$	$\alpha_{n,1,n}$	$\alpha_{n,n,1}$	$\alpha_{n,n,2n-1}$	$\alpha_{n,2n-1,n}$	$\alpha_{n,n,n}$

Table 1.

Let for every natural number $n \geq 2$:

$$\gamma_1 = \alpha_{1,1,1},$$

$$\gamma_2 = \alpha_{2,2,2},$$

$$\gamma_3 = \alpha_{3,3,3},$$

$$\ldots$$

The α-elements that correspond to points over a pyramid's face are calculated by formulas for a standard PLT from [1].

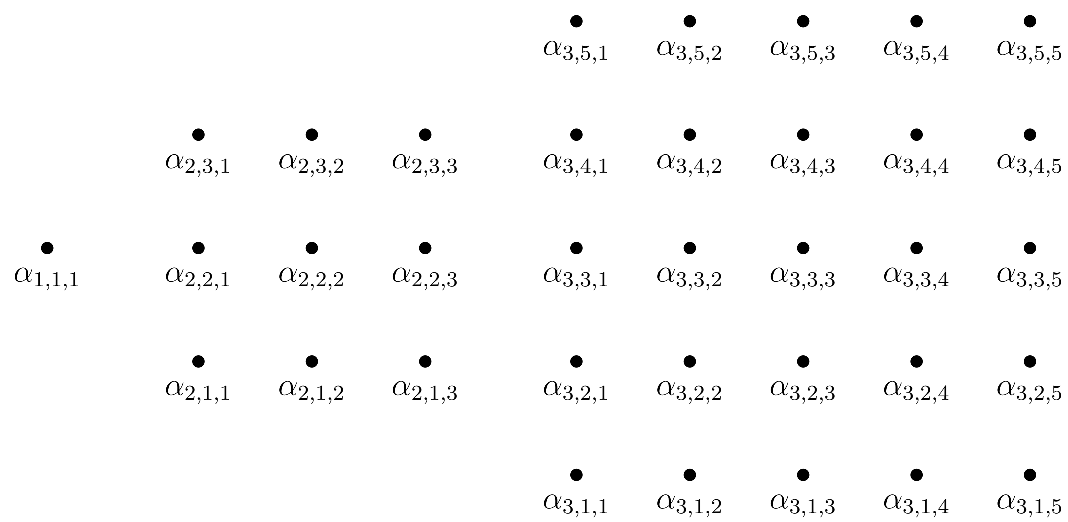

Figure 3.

In [4], we discussed the case that all the four A-lines coincide. Therefore, all the four B-lines coincide.

Let $\{b_i\}_{i\geq 1}$ and $\{c_i\}_{i\geq 1}$ be two (infinite) sequences. A more general case than the one above is the following:

$$\alpha_{i,1,1} = \alpha_{i,2i-1,2i-1} = b_i,$$

$$\alpha_{i,1,2i-1} = \alpha_{i,2i-1,1} = c_i.$$

Therefore, we can prove, e.g., by induction the validity of

Lemma. For every natural number $i \geq 1$:

$$\alpha_{i,1,i} = \alpha_{i,i,1} = \alpha_{i,i,2i-1} = \alpha_{i,2i-1,i}.$$

The most general case is, when four different (infinite) sequences $\{2b_i\}_{i\geq 1}$, $\{2c_i\}_{i\geq 1}$, $\{2d_i\}_{i\geq 1}$ and $\{2e_i\}_{i\geq 1}$ are given.

Let a be a fixed real number and let for every $i \geq 1$

$$\psi_i = b_i + c_i + d_i + e_i.$$

The elements in the first three levels of the pyramid are the following:

first level

a

second level

$2e_1$	$a + d_1 + e_1$	$2d_1$
$a + b_1 + e_1$	$2a + \psi_1$	$a + c_1 + d_1$
$2b_1$	$a + b_1 + c_1$	$2c_1$

third level

$2e_2$	$2e_1 + 2e_2$	$a + 2d_1 + 2e_1$ $+d_2 + e_2$	$2d_1 + 2d_2$	$2d_2$
$2e_1 + 2e_2$	$2e_1 + 2e_2$	$2a + 3d_1 + 3e_1$ $d_2 + e_2$	$2d_1 + 2d_2$	$2d_1 + 2d_2$
$a + 2b_1 + 2e_1$ $b_2 + e_2$	$2a + 3b_1 + 3e_1$ $b_2 + e_2$	$4a + 4\psi_1 + \psi_2$	$2a + 3c_1 + 3d_1$ $c_2 + d_2$	$a + 2c_1 + 2d_1$ $c_2 + d_2$
$2b_1 + 2b_2$	$2b_1 + 2b_2$	$2a + 3b_1 + 3c_1$ $b_2 + c_2$	$2c_1 + 2c_2$	$2c_1 + 2c_2$
$2b_2$	$2b_1 + 2b_2$	$a + 2b_1 + 2c_1$ $+b_2 + c_2$	$2c_1 + 2c_2$	$2c_2$

It is seen easy, that for every natural numbers n and $1 \leq i \leq n$:

$$a_{n,n,i} + a_{n,n,2n-i} = a_{n,i,n} + a_{n,2n-i,n}.$$

The first seven members of sequence $\{\gamma_i\}_{i\geq 1}$ are:

$$\gamma_1 = a,$$

$$\gamma_2 = 2a + \psi_1,$$

$$\gamma_3 = 4a + 4\psi_1 + \psi_2,$$

$$\gamma_4 = 8a + 12\psi_1 + 6\psi_2 + \psi_3,$$

$$\gamma_5 = 16a + 32\psi_1 + 24\psi_2 + 8\psi_3 + \psi_4,$$

$$\gamma_6 = 32a + 80\psi_1 + 80\psi_2 + 40\psi_3 + 10\psi_4 + \psi_5,$$

$$\gamma_7 = 64a + 192\psi_1 + 240\psi_2 + 160\psi_3 + 60\psi_4 + 12\psi_5 + \psi_6.$$

By induction, we can prove the following

Lemma. If for some natural number $n \geq 2$ and for a generating sequence $\{x_i\}_{i\geq 1}$:

$$\gamma_n = \varphi_{n,0}a + \sum_{i=1}^{n} \varphi_{n,i}x_i,$$

then

$$\varphi_{n,0} = 2^{n-1},$$

$$\varphi_{n,i} = \varphi_{n-1,i-1} + 2\varphi_{n-1,i}, \quad \text{for } 1 \leq i \leq n-1,$$

$$\varphi_{n,n-1} = 1.$$

Six-, eight-, etc., faced Pascal's pyramids can be constructed by the above ways. By this reason, here will give only the ideas about their forms, showing only the second level of them, while in all cases, the first level contains only element a.

The second level of the simplest case of the 6-face Pascal's pyramid has the form

$$\begin{array}{ccccccc}
 & & b & a+b & b & & \\
 & a+b & & & & a+b & \\
b & & & 2a+b & & & b \\
 & a+b & & & & a+b & \\
 & & b & a+b & b & &
\end{array}$$

The second level of the more complex case of the 6-face Pascal's pyramid has the form

$$\begin{array}{ccccccc} & & 2b_1 & a+b_1+c_1 & 2c_1 & & \\ & a+b_1+c_1 & & & & a+b_1+c_1 & \\ 2c_1 & & & 2a+b_1+c_1 & & & 2b_1 \\ & a+b_1+c_1 & & & & a+b_1+c_1 & \\ & & 2b_1 & a+b_1+c_1 & 2c_1 & & \end{array}$$

Finally, the second level of the most complex case of the 6-face Pascal's pyramid has the form

$$\begin{array}{ccccccc} & & 2b_1 & a+b_1+c_1 & 2c_1 & & \\ & a+b_1+g_1 & & & & a+c_1+d_1 & \\ 2g_1 & & & 2a+\varphi_1 & & & 2d_1 \\ & a+f_1+g_1 & & & & a+d_1+e_1 & \\ & & 2f_1 & a+e_1+f_1 & 2e_1 & & \end{array}$$

Here,

$$\varphi_1 = \frac{1}{2}(b_1+c_1+e_1+f_1) = \frac{1}{2}(c_1+d_1+f_1+g_1) = \frac{1}{2}(b_1+d_1+e_1+g_1).$$

Therefore,

$$b_1+e_1 = d_1+g_1, \;\; c_1+f_1 = d_1+g_1, \;\; c_1+f_1 = b_1+e_1.$$

In general case, for every $i \geq 1$

$$b_i+e_i = d_i+g_i, \;\; c_i+f_i = d_i+g_i, \;\; c_i+f_i = b_i+e_i.$$

Similarly, we can construct the 8-face Pascal's pyramids. Here, we show only the second level of the most complex case.

$$
\begin{array}{ccccccc}
 & & 2b_1 & a+b_1+c_1 & 2c_1 & & \\
 & a+b_1+k_1 & & & & a+c_1+d_1 & \\
2k_1 & & & & & & 2d_1 \\
a+h_1+k_1 & & & 2a+\varphi_1 & & & a+d_1+e_1 \\
2h_1 & & & & & & 2e_1 \\
 & a+g_1+h_1 & & & & a+e_1+f_1 & \\
 & & 2g_1 & a+f_1+g_1 & 2f_1 & &
\end{array}
$$

Here, the necessity condition that must be valid is

$$
\begin{aligned}
\varphi_1 &= \frac{1}{2}(b_1+c_1+f_1+g_1) = \frac{1}{2}(c_1+d_1+g_1+h_1) \\
&= \frac{1}{2}(d_1+e_1+h_1+k_1) = \frac{1}{2}(b_1+e_1+f_1+k_1).
\end{aligned}
$$

In general case, for every $i \geq 1$

$$
\begin{aligned}
b_i+c_i+f_i+g_i &= c_i+d_i+g_i+h_i \\
= d_i+e_i+h_i+k_i &= b_i+e_i+f_i+k_i.
\end{aligned}
$$

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