Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 21, 2015, No. 2, 1–7

On the inequalities for beta function

Barkat Ali Bhayo[1] and József Sándor[2]

[1] Department of Mathematical Information Technology
University of Jyväskylä, 40014 Jyväskylä, Finland
e-mail: bhayo.barkat@gmail.com

[2] Department of Mathematics, Babeş-Bolyai University
Str. Kogalniceanu nr. 1, 400084 Cluj-Napoca, Romania
e-mail: jsandor@math.ubbcluj.ro

Abstract: Here authors establish the sharp inequalities for classical beta function by studying the inequalities of trigonometric sine function.
Keywords: Gamma function, Beta function, Sine function, Jordan inequality.
AMS Classification: 33B15, 26D05, 26D07, 26D15.

1 Introduction

For $x, y > 0$, the classical *gamma function* Γ, the *digamma function* ψ and the *beta function* $B(\cdot,\cdot)$ are defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively. These functions have important applications in various branches of science and engineering [7]. Since the 1950s, numerous authors have given several inequalities involving these functions by employing different approaches, see for example [3, 4, 8, 13]. In this paper, we establish the inequalities for beta function by using the well-known Jordan's inequality [10, 11, 12].

The functions Γ and ψ satisfy the following recurrence relations

$$\Gamma(1+x) = x\Gamma(x), \quad \psi(1+x) = \frac{1}{x} + \psi(x). \tag{1}$$

Weierstrass expressed the gamma and sine functions in terms of innite products as follows

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/e}, \quad \sin(\pi x) = \pi x\prod_{n\neq 0}^{\infty}\left(1-\frac{x}{n}\right)e^{x/n},$$

where γ is the Euler–Mascheroni constant [1] defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) = 0.57721566\ldots.$$

These definitions give the following relation

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}, \quad t \notin \mathbb{Z}, \tag{2}$$

which is known as the Euler's reflection formula [1, 6.1.17]. We refer the reader to [8] for a historical background and properties of the gamma and beta functions.

Dragomir et al. [8] established the following inequality

$$B(x,y) \le \frac{1}{xy}, \quad x,y \in (0,1). \tag{3}$$

Alzer [2] established the following inequalities with right side refining relation (3)

$$\frac{1}{xy}\left(1 - a\frac{1-x}{1+x}\frac{1-y}{1+y}\right) < B(x,y) \tag{4}$$

$$< \frac{1}{xy}\left(1 - b\frac{1-x}{1+x}\frac{1-y}{1+y}\right), \quad x,y \in (0,1),$$

with the best possible constants $a = 2\pi^2/3 - 4 \approx 2.57973$ and $b = 1$. Recently, the second inequality in (4) was refined by Ivády [9]

$$\frac{1}{xy}(x + y - xy) \le B(x,y) \le \frac{1}{xy}\frac{x+y}{1+xy}, \quad x,y \in (0,1). \tag{5}$$

Lemma 1. *Let f be a twice differentiable function on $(0,\pi)$, and let $g(x) = f(x)/\sin(x)$, $h(x) = g'(x)\sin(x)^2$, and $F(x) = f(x) + f''(x)$. Then $h'(x)$ and $F(x)$ have the same sign on $(0,\pi)$.*

Proof. Straightforward. □

Theorem 1. *For $t \in (0,1)$, we have*

$$\frac{3(1-t)}{\pi t^2 - \pi t + \pi} < \frac{\sin(\pi t)}{\pi t} < \frac{\pi(1-t)}{\pi t^2 - \pi t + \pi}, \tag{6}$$

$$1 - (2-t)t^2 < \frac{\sin(\pi t)}{\pi t} < \frac{16}{5\pi}\left(1 - (2-t)t^2\right). \tag{7}$$

Proof. Let $g(x) = f(x)/\sin(x)$ for $x \in (0,\pi)$, where

$$f(x) = (\pi x - x^2)/(\pi^2 - \pi x + x^2).$$

We get

$$\frac{(\pi^2 - \pi x + x^2)^3}{x(\pi - x)} F(x) = (\pi^2 - \pi x + x^2)^2 - 6\pi^2 = A(x)B(x),$$

where $B(x) > 0$ always, and $A(x) = x^2 - \pi x + \pi^2 - \pi\sqrt{6}$.

The roots of equation $A(x) = 0$ are $x_1 = (\pi - \sqrt{4 \cdot 6^{1/2}\pi - \pi^2})/2$ which is in $(0, \pi/2)$, and $x_2 = (\pi + \sqrt{4 \cdot 6^{1/2}\pi - \pi^2})/2$ which is in $(\pi/2, \pi)$. Let $x \in (0, x_1)$, then $A(x) > 0$, so $F(x) > 0$, giving $h'(x) > 0$, where F and h are as in Lemma 1. This implies $h(x) > h(0) = 0$, so $g'(x) > 0$ by Lemma 1.

Again, let $x \in [x_1, \pi/2)$, then $A(x) \geq 0$, giving $F(x) \leq 0$, i.e. $h'(x) \leq 0$. This implies $h(x) > h(\pi/2) = 0$. So $g'(x) \geq 0$ here too. We have proved that $g'(x) > 0$ for all x in $(0, \pi/2)$. Let now x in $(\pi/2, x_2)$. Then $A(x) < 0$, so $h'(x) < 0$, implying $h(x) < h(\pi/2) = 0$. For x in $[x_2, \pi)$ one has $h'(x) \geq 0$, so $h(x) \leq h(\pi) = 0$. Therefore, for all x in $(\pi/2, \pi)$ one has $h(x) < 0$, i.e. $g'(x) < 0$ here. In both cases we had $g'(x) = 0$ only for $x = \pi/2$. Consequently, the function g is strictly increasing in $(0, \pi/2)$ and strictly decreasing in $(\pi/2, \pi)$, and attains maximum $1/3$ at $x = \pi/2$ as well as g tends to $1/\pi$ when x tends to 0 or π. This implies the proof of (6) if we let $x = \pi t$.

For the proof of (7), let

$$f(x) = \frac{\sin(x)}{x(x^3 - 2\pi x^2 + \pi^3)}.$$

A simple calculation gives

$$(x\,k(x))^2 f'(x) = g(x),$$

where $k(x) = x^3 - 2\pi x^2 + \pi^3$ and

$$g(x) = \cos(x).(x^4 - 2\pi x^3 + \pi^3 x) - \sin(x).(4x^3 - 6\pi x^2 + \pi^3).$$

It is immediate that $g(0) = g(\pi/2) = g(\pi) = 0$. One has $g'(x) = -x\sin(x)h(x)$, with

$$h(x) = x^3 - 2\pi x^2 + 12x + \pi^3 - 12\pi.$$

Here $h(0) = \pi(\pi^2 - 12) < 0$, $h(\pi/2) = 5\pi^3/8 - 6\pi > 0$ as $5\pi^2 > 48$ and $h(\pi) = 0$. Further $h'(x) = 3x^2 - 4\pi x + 12$, and $h''(x) = 2(3x - 2\pi)$. The roots of $h'(x) = 0$ are $x_1 = (2\pi - 2\sqrt{\pi^2 - 9})/3 \approx 1.47271$, which is in $(0, \pi/2)$, and $x_2 = (2\pi + 2\sqrt{\pi^2 - 9})/3 \approx 2.71608$, which is in $(2\pi/3, \pi)$. Therefore, $h(x)$ is strictly increasing in $(0, x_1)$, and (x_2, π), while strictly decreasing in (x_1, x_2).

Let $x \in (0, \pi/2)$, then as $h(0) < 0$, $h(\pi/2) > 0$, h has a single root x_0, and a maximum point in x_1. Thus $h(x) < 0$ in $(0, x_0)$, and $h(x) > 0$ in $(x_0, \pi/2)$. Therefore, $g'(x) > 0$ for $x \in (0, x_0)$ and $g'(x) < 0$ in $(x_0, \pi/2)$. Thus $g(x) > g(0) = 0$ in $(0, x_0)$ and $g(x) > g(\pi/2) = 0$ $(x_0, \pi/2)$. In all cases, $g(x) > 0$ for $x \in (0, \pi/2)$. This means that, $f(x)$ is strictly increasing in $(0, \pi/2)$.

When x is in $(\pi/2, \pi)$, the proof runs as above, by remarking that by $h(2\pi/3) < 0$, there exists a unique $x_0^* \in (\pi/2, \pi)$ such that $h(x_0^*) = 0$. Since $h(x) > 0$ in (x_0^*, π) and $h(x) < 0$ in (x_0^*, π) we get that $g(x) < g(\pi/2) = 0$ in $(\pi/2, x_0^*)$, while $g(x) < g(\pi) = 0$ in (x_0^*, π), so in all cases $g(x) < 0$, when x is in $(\pi/2, \pi)$. Thus $f(x)$ is strictly decreasing in $(\pi/2, \pi)$. This completes the proof. □

The inequalities in (6) and (7) are not comparable. From the proof of (7) we get the following corollary.

Corollary 1. *For $x \in (0, \pi/2)$, we have*

$$\frac{x^3 - 2\pi x^2 + \pi^3}{4x^3 - 6\pi x^2 + \pi^3} > \frac{\tan(x)}{x},$$

inequality reverses for $x \in (\pi/2, \pi)$.

Theorem 2.

$$B(x, y) < \frac{1}{xy}\frac{x+y}{1+xy}, \quad x, y \in (0, 1),$$

inequality reverses for $x > 1$.

Proof. See [6, Theorem 1.28, p. 18]. □

Corollary 2. *We have*

$$\frac{\alpha}{xy}\frac{x+y}{1+xy} < B(x, y) < \frac{\beta}{xy}\frac{x+y}{1+xy}, \quad x \in (0, 1) \text{ with } y = 1 - x,$$

with the best possible constants $\alpha = 5\pi/16 \approx 0.98175$ and $\beta = 1$.

Proof. Utilizing (2), the first inequality in (7) can be written as

$$t(1-t)(1+t(1-t)) < \frac{1}{\Gamma(t)\Gamma(1-t)},$$

which is equivalent to

$$\frac{t(1-t)(1+t(1-t))}{t+1-t} < \frac{\Gamma(t+1-t)}{\Gamma(t)\Gamma(1-t)}.$$

Letting $x = t$ and $y = 1 - t$, we get the first inequality. The second inequality follows similarly from the second inequality of (7). This completes the proof. □

Remark 1. *The inequality*

$$1 - \frac{z}{\pi} < \frac{\sin(z)}{z}, \quad z \in (0, \pi),$$

can be written as

$$\Gamma\left(1 + \frac{z}{\pi}\right)\Gamma\left(1 - \frac{z}{\pi}\right) < \frac{1}{1 - z/\pi},$$

by (2). *This implies* (3) *if we let $x = z/\pi$ and $y = 1 - z/\pi$.*

Lemma 2. *We have*

$$\psi(1+x) - \psi(x+y) < \frac{1-y}{x+y-xy}, \quad x > 1,\ y \in (0, 1), \tag{8}$$

$$\psi(2-x) - \psi(1+x) < \frac{1-2x}{1-(1-x)x}, \quad x \in (0, 1/2), \tag{9}$$

inequality reverses for $x \in (1/2, 1)$.

Proof. For $x > 1$ and $y \in (0,1)$, we define

$$g_x(y) = \psi(1+x) - \psi(x+y) - \frac{1-y}{x+y-xy}.$$

Differentiating with respect to y we get

$$\begin{aligned} g_x''(y) &= -\frac{2(1-x)^2(1-y)}{(x(1-y)+y)^3} - \frac{2(1-x)}{(x(-y)+x+y)^2} - \psi''(x+y) \\ &= \frac{2x-2}{(x(1-y)+y)^3} - \psi''(x+y) > 0, \end{aligned}$$

since $\psi''(x+y) < 0$. Thus, g_x is convex in y, clearly $g_x(0) = g_x(1) = 0$. This implies that the graph of the function g_x on $(0,1)$ lies under the line segment joining origin and the point $(1,0)$. The proof is now obvious.

For (9), write

$$f(x) = \psi(2-x) - \psi(1+x) - \frac{1-2x}{1-(1-x)x}, \quad x \in (0,1).$$

One has

$$\begin{aligned} f''(x) &= \left(\frac{2(2x-1)^2}{(1-(1-x)x)^3} - \frac{2}{(1-(1-x)x)^2}\right)(2x-1) \\ &\quad -\frac{4(2x-1)}{(1-(1-x)x)^2} + \psi''(2-x) - \psi''(x+1) \\ &= \frac{2(x-2)(x+1)(2x-1)}{((x-1)x+1)^3} + \psi''(2-x) - \psi''(x+1). \end{aligned}$$

Clearly, the function ψ'' is increasing and negative. So, it is not difficult to see that f'' is positive for $x \in (0,1/2)$, and negative for $x \in (1/2,1)$. This implies the convexity and concavity of f in $x \in (0,1/2)$ and $x \in (1/2,1)$, respectively. Clearly, $f(0) = f(1/2) = f(1) = 0$. This completes the proof. □

Theorem 3. *We have*

$$\frac{1}{xy}(x+y-xy) > B(x,y), \quad x > 1,\, y \in (0,1),$$

inequality reverses for $x \in (0,1)$.

Proof. The proposed inequality can be written as

$$h_y(x) = \log(\Gamma(1+x)) + \log(\Gamma(1+y)) - \log(\Gamma(x+y)) + \log(x+y-xy) > 0.$$

Clearly, $h_y(1) = 0$. Differentiation with respect to x yields

$$h_y'(x) = \psi(1+x) - \psi(x+y) - \frac{1-y}{x+y-xy} = g_x(y),$$

which is negative by (8). Thus the function $h_y(x)$ is decreasing in $x > 1$, this gives the desired inequality. □

Corollary 3. *We have*

$$B(x,y) < \frac{\pi}{3}\frac{1}{xy}(x+y-xy), \quad x \in (0,1), \text{ with } y = 1-x.$$

Proof. Let

$$h(x) = \log\left(\frac{1}{3}\pi(1-(1-x)x)\right) - \log(\Gamma(2-x)) - \log(\Gamma(x+1)),$$

clearly $h(1/2) = 0$. One has,

$$h'(x) = \psi(2-x) - \psi(1+x) - \frac{1-2x}{1-(1-x)x},$$

which is positive in $x \in (0,1/2)$ and negative in $x \in (1/2,1)$ by (9). This implies that h is decreasing in $x \in (0,1/2)$ and increasing in $x \in (1/2,1)$. Thus the proof follows. □

References

[1] Abramowitz, M., & Stegun, I., eds. (1965) *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* National Bureau of Standards, Dover, New York.

[2] Alzer, H. (2003) Some beta function inequalities, *Proc. of the Royal Soc. of Edinburgh*, 133A, 731–745.

[3] Alzer, H. (1993) Some gamma function inequalities, *Math. Comp.*, 60, 337–346.

[4] Alzer, H. (1997) On some inequalities for the gamma and psi functions, *Math. Comp.*, 66, 373–389.

[5] Anderson, G.D., Vamanamurthy, M.K., & Vuorinen, M. (1993) Inequalities of quasiconformal mappings in the space, *Pacific J. Math.*, 160(1).

[6] Bhayo, B.A., & Sándor, J. (2014) On classical inequalities of trigonometric and hyperbolic functions, May 2014, `http://arxiv.org/pdf/1405.0934.pdf`.

[7] Andrews, G., Askey, R., & Roy, R. (1999) *Special Functions, Encyclopedia of Mathematics and its Applications*, Vol. 71, Cambridge Univ. Press.

[8] Dragomir, S.S., Agarwal, R.P. & Barnett, N.S. (2000) Inequalities for beta and gamma functions via some classical and new integral inequalities, *J. Inequal. Appl.*, 5, 103–165.

[9] Ivády, P. (2012) On a beta function inequality, *J. Math. Inequal.*, 6(3), 333–341.

[10] Klen, R., Visuri, M., & Vuorinen, M. (2010) On Jordan type inequalities for hyperbolic functions, *J. Ineq. Appl.*, Vol. 2010, Art. ID 362548, pp. 14.

[11] Mitrinović, D.S. (1970) *Analytic Inequalities*, Springer-Verlag, Berlin.

[12] Neuman, E. & Sándor, J. (2010) On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, *Math. Inequal. Appl.* 13(4), 715–723.

[13] Qiu, S.-L. & Vuorinen, M. (2004) Some properties of the gamma and psi functions with applications, *Math. Comp.*, 74(250), 723–742.

[14] Sándor, J. (2014) A bibliography on gamma functions: inequalities and applications, June 2014, Available online: `http://www.math.ubbcluj.ro/jsandor/letolt/art42.pdf`.

[15] Spanier, J. & Oldham, K.B. (1987) *An atlas of functions*, Hemisphere Publishing, Washington.