Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 21, 2015, No. 1, 86–91

Some identities on Schläfli-type mixed modular equations

B. R. Srivatsa Kumar[1], B. Sowmya Navada[2] and Ranjani Nayak[2]

[1] Department of Mathematics, Manipal Institute of Technology
Manial University, Manipal – 576 104, India
e-mail: sri_vatsabr@yahoo.com

[2] Department of Sciences, Manipal University
Manipal – 576 104, India

Abstract: Srinivasa Ramanujan has recorded eleven Schläfli-type mixed modular equations on p. 86 of his first notebook. Using some of these in this paper, we obtain some new identities of the same type by using Maple.

Keywords: Theta-functions, Modular equations.
AMS Classification: Primary: 11B65, 33D10; Secondary: 11F27.

1 Introduction

In Chapter 16 of his second notebook [3], Ramanujan developed the theory of theta function and his theta function is defined by

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab|<1.$$

Following Ramanujan, we define

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.1}$$

and

$$\chi(q) := (-q;q^2)_\infty,$$

A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;x):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n, \quad |x|<1,$$

is the generalized hypergeometric series with

$$(a)_n := a(a+1)(a+2)...(a+n-1).$$

Then, we say that β is of n-th degree over α and call the ratio

$$m := \frac{z_1}{z_n}$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

L. Schläfli [6] established certain identities, which provides relations between P and Q, where

$$P = 2^{1/6}[\alpha\beta(1-\alpha)(1-\beta)]^{1/24} \qquad \text{and} \qquad Q = \left[\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right]^{1/24},$$

for β having degrees $3, 5, 7, 11, 13, 17$ and 19 respectively over α. Ramanujan recorded eleven Schläfli-type mixed modular equations in his first notebook [5, pp. 86–88]. B. C. Berndt [4, pp. 379–384], employed the theory of modular forms to prove these results.

N. D. Baruah [1] proved two of these equations by deriving some theta-function identities from Schroter's formulae. Recently, Baruah [2] proved seven of these on employing Ramanujan's theta-function identities, Schläfli-type modular equations and Russell-type and Weber-type modular equations. In the process, he also obtained three new Schläfli-type mixed modular equations. Recently, K. R. Vasuki and T. G. Sreeramamurthy [7] have established certain new Ramanujan's Schläfli-type mixed modular equations, by employing Ramanujan's modular equations and Schläfli modular equations.

In Section 2 of this paper, we list Schläfli type modular equations and in Section 3, we prove four new identities of the same type.

2 Preliminary results

Now we state the Ramanujan's mixed modular equations. We set

$$P := \left(256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\right)^{1/48}, \tag{2.1}$$

$$Q := \left(\frac{a\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/48}, \tag{2.2}$$

$$R := \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/48}, \tag{2.3}$$

$$T := \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/48}. \tag{2.4}$$

Theorem 2.1. *If* α,β,γ,*and* δ *are of degrees* 1, 5, 7 *and* 35*, respectively, then*

$$Q^4 + \frac{1}{Q^4} - \left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0. \tag{2.5}$$

Theorem 2.2. *If* α, β, γ, δ *are of degrees* 1, 5, 7, *and* 35*, respectively, then*

$$R^8 + \frac{1}{R^8} + 7\left(3 + 2\left(P^2 + \frac{1}{P^2}\right)\right)\left(R^4 + \frac{1}{R^4}\right) - 8\left(P^6 + \frac{1}{P^6}\right) + 28\left(P^2 + \frac{1}{P^2}\right) + 56 = 0. \tag{2.6}$$

Theorem 2.3. *If* α,β,γ,*and* δ *are of degrees* 1, 5, 7 *and* 35*, respectively, then*

$$R^4 + \frac{1}{R^4} - \left(Q^6 + \frac{1}{Q^6}\right) + 5\left(Q^4 + \frac{1}{Q^4}\right) - 10\left(Q^2 + \frac{1}{Q^2}\right) + 15 = 0. \tag{2.7}$$

Theorem 2.4. *If* α,β,γ,*and* δ *are of degrees* 1, 3, 7 *and* 21*, respectively, then*

$$R^8 + \frac{1}{R^8} + 7\left(R^6 + \frac{1}{R^6}\right) + 14\left(R^4 + \frac{1}{R^4}\right) + 21\left(R^2 + \frac{1}{R^2}\right) - 8\left(P^6 + \frac{1}{P^6}\right) + 42 = 0. \tag{2.8}$$

Theorem 2.5. *If* α,β,γ,*and* δ *are of degrees* 1,3,7 *and* 21*, respectively, then*

$$Q^{16} + \frac{1}{Q^{16}} - 5\left(Q^{12} + \frac{1}{Q^{12}}\right) + 5\left(Q^8 + \frac{1}{Q^8}\right) + 6\left(Q^4 + \frac{1}{Q^4}\right) - 8\left(P^6 + \frac{1}{P^6}\right) + 6 = 0. \tag{2.9}$$

Theorem 2.6. *If* α,β,γ,*and* δ *are of degrees* 1,5,7 *and* 35*, respectively, then*

$$T^6 + \frac{1}{T^6} + 5\sqrt{2}\left(T^3 + \frac{1}{T^3}\right)\left(P + \frac{1}{P}\right) - 4\left(P^4 + \frac{1}{P^4}\right) + 10 = 0. \tag{2.10}$$

3 Main results

Theorem 3.1. *If* α, β, γ, δ *are of degrees* 1, 5, 7, *and* 35*, respectively, then we have*

$$R^8 + \frac{1}{R^8} - \left(Q^{12} + \frac{1}{Q^{12}}\right) + 3\left(Q^{10} + \frac{1}{Q^{10}}\right) - 3\left(Q^8 + \frac{1}{Q^8}\right) + 4\left(Q^6 + \frac{1}{Q^6}\right) + 17\left(Q^4 + \frac{1}{Q^4}\right)$$
$$-17\left(Q^2 + \frac{1}{Q^2}\right) + 7\left(R^4 + \frac{1}{R^4}\right)\left\{\left(Q^4 + \frac{1}{Q^4}\right) - \left(Q^2 + \frac{1}{Q^2}\right) + 3\right\} + 50 = 0.$$

Proof. On setting,

$$R^4+\frac{1}{R^4}=x,\qquad P^2+\frac{1}{P^2}=y,\qquad Q^2+\frac{1}{Q^2}=z$$

in (2.5) and (2.6), we have

$$z^2-z-2y-2=0.$$

and

$$8y^3-x^2-21x-14xy-52y-54=0$$

On eliminating y between the above using Maple, and on simplyfying we have the result. □

Theorem 3.2. *If* α, β, γ, δ *are of degrees* 1,5, 7, *and* 35, *respectively, then we have*

$$8\left(P^6+\frac{1}{P^6}\right)-28\left(P^2+\frac{1}{P^2}\right)-14\left(P^2+\frac{1}{P^2}\right)\left(R^4+\frac{1}{R^4}\right)$$
$$-\left(R^8+\frac{1}{R^8}\right)-21\left(R^4+\frac{1}{R^4}\right)-56=0.$$

Proof. On setting,

$$R^4+\frac{1}{R^4}=x,\qquad P^2+\frac{1}{P^2}=y,\qquad Q^2+\frac{1}{Q^2}=z$$

(2.5) and (2.7) reduces to

$$z^2-z-2y-2=0$$

and

$$x-z^3+5z^2-7z+5=0.$$

On eliminating z between the above using Maple, we have

$$8y^3-x^2-21x-14xy-52y-54=0,$$

and on expanding this, we have the result. □

Theorem 3.3. *If* α,β,γ,*and* δ *are of degrees* 1,3,7 *and* 21, *respectively, then*

$$\left(R^8+\frac{1}{R^8}\right)+7\left(R^6+\frac{1}{R^6}\right)+14\left(R^4+\frac{1}{R^4}\right)+21\left(R^2+\frac{1}{R^2}\right)-\left(Q^{16}+\frac{1}{Q^{16}}\right)$$
$$+5\left(Q^{12}+\frac{1}{Q^{12}}\right)-5\left(Q^8+\frac{1}{Q^8}\right)-6\left(Q^4+\frac{1}{Q^4}\right)=0.$$

Proof. On setting,

$$R^2+\frac{1}{R^2}=x,\qquad P^6+\frac{1}{P^6}=y,\qquad Q^4+\frac{1}{Q^4}=z.$$

(2.8) and (2.9) reduces to

$$x^4 + 7x^3 + 10x^2 - 8y - 26 = 0$$

and

$$z^4 - 5z^3 + z^2 + 21z - 8y - 8 = 0.$$

On eliminating y between the above using Maple, we get the equation

$$x^4 + 7x^3 + 10x^2 - z^4 + 5z^3 - z^2 + 21z + 18 = 0$$

and on expanding this, we have the result. □

Theorem 3.4. *If* α,β,γ,*and* δ *are of degrees* 1, 5, 7 *and* 35*, respectively, then*

$$3\left(R^4+\frac{1}{R^4}\right)^3-\left(R^4+\frac{1}{R^4}\right)^4+10\left(R^4+\frac{1}{R^4}\right)^2-28\left(R^4+\frac{1}{R^4}\right)$$
$$-4\left(R^4+\frac{1}{R^4}\right)\left(T^3+\frac{1}{T^3}\right)^2+60\left(T^3+\frac{1}{T^3}\right)^2+\left(R^4+\frac{1}{R^4}\right)\left(T^3+\frac{1}{T^3}\right)^4$$
$$-14\left(T^3+\frac{1}{T^3}\right)^4+\left(T^3+\frac{1}{T^3}\right)^6-56=0.$$

Proof. On setting,

$$P+\frac{1}{P}=y, \qquad T^3+\frac{1}{T^3}=a, \qquad R^4+\frac{1}{R^4}=x$$

then (2.6) and (2.10) reduces to

$$x^2 - 7x - 8y^6 + 48y^4 - 44y^2 + 14xy^2 + 14 = 0$$

and

$$a^2 + 5\sqrt{2}ay - 4y^4 + 16y^2 = 0.$$

On eliminating y between the above using Maple, weobtain

$$(3x^3 + 10x^2 - x^4 - 28x - 56 - x^2a^2 - 4xa^2 + 60a^2 + xa^4 - 14a^4 + a^6)$$

$$\times(-x^4-87x^3-1565x^2+14147x-30926+174a^2x^2-2089xa^2+2650a^2-29a^4x-294a^4+a^6) = 0.$$

From the definition of R and T as $q \to 0$, the first factor vanishes for q sufficiently small, whereas the second factor doesnot. By the identity theorem, the first factor vanishes for $|q| < 1$. Finally on expanding this, we have the result. □

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