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GCED reciprocal LCEM matrices

Zahid Raza and Seemal Abdul Waheed

Department of Mathematics
National University of Computer & Emerging Sciences
B-Block, Faisal Town, Lahore, Pakistan
e-mails: zahid.raza@nu.edu.pk, seemalw11@gmail.com

Abstract: We have given structure theorems for a greatest common exponential divisor (GCED) and reciprocal least common exponential multiple (LCEM) matrix and calculated their determinants. The inverses and determinants of GCED and reciprocal LCEM matrices on exponential divisor closed sets have been determined.

Keywords: GCED matrix, Reciprocal LCEM matrix, Exponential divisor, Exponential divisor closed set.
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1 Introduction

In 1876, Smith [9] proved that the determinant of a gcd matrix on $S = \{1, 2, \ldots, n\}$ is equal to $\varphi(1)\varphi(2)\ldots\varphi(n)$ where φ is Euler's totient function. The result holds if S is a factor closed set. Structure theorems for reciprocal gcd matrices and lcm matrices were introduced by Beslin [2]. Structures of Power gcd matrix, Power lcm matrix, reciprocal lcm matrix, gcd reciprocal lcm matrix, gcud reciprocal lcum matrices have been determined [1, 3, 5, 7, 8, 10, 12]. Research has also been extended to divisibility properties of such matrices and their applications [6, 4].

We recall that an integer $d = \prod_{i=1}^{t} {p_i}^{a_i}$ is said to be an exponential divisor of $m = \prod_{i=1}^{t} {p_i}^{b_i}$, if $a_i | b_i$ for every $1 \leq i \leq t$ and is denoted by $d|_e m$. This notion was introduced by Subarrao [11]. Note that unlike divisor and unitary divisor, 1 is not an exponential divisor for every $m > 1$. The smallest exponential divisor of $m > 1$ is its square free kernel $\kappa(m) = \prod_{i=1}^{r} p^i$ [13].

Two integers n and m have common exponential divisor if and only if they have the same prime factors. Two integers $m = \prod_{i=1}^{r} {p_i}^{b_i}$ and $n = \prod_{i=1}^{r} {p_i}^{c_i}$ are exponentially co-prime if $(b_i, c_i) = 1$

for every $1 \leq i \leq r$. We denote the GCED of two integers m and n by $(m, n)_e$. By convention $(1, 1)_{(e)} = 1$ and $(1, m)_{(e)}$ does not exist for every $m > 1$.

A set $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is said to be an exponential divisor closed set if the exponential divisors of every element of S belongs to S. For example $\{12, 18, 36\}$ is not an exponential divisor closed set. But, $\{6, 12, 18, 36\}$ is an exponential divisor closed set.

Similarly, a set $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is said to be GCED closed if $(x_i, x_j)_{(e)} \in S$ for every $x_i, x_j \in S$. Note that $\{6, 12, 18, 36\}$ is also a GCED closed set. It should be noted that in this paper only those sets have been considered for which the GCED or LCEM of every pair of numbers exists.

The exponential convolution of two arithmetic functions f and g is given as

$$(f \odot g)(n) = \sum_{k_1 l_1 = m_1} \cdots \sum_{k_r l_r = m_r} f({p_1}^{k_1} {p_2}^{k_2} \ldots {p_r}^{k_r}) g({p_1}^{l_1} {p_2}^{l_2} \ldots {p_r}^{l_r})$$

where $n = {p_1}^{m_1} {p_2}^{m_2} \ldots {p_r}^{m_r}$.

The inverse with respect to $\odot$ of the constant function 1 is called the exponential analogue of Möbius function and is denoted by $\mu^{(e)}$.

2 Structure of GCED reciprocal LCEM matrix

Definition 1. *Let $T = \{x_1, x_2, x_3, \ldots, x_n\}$ be an ordered set of positive integers greater than 1. The $n \times n$ matrix $(T)_{(e)} = (t_{ij})_{(e)}$ having $t_{ij} = \frac{(x_i, x_j)_{(e)}}{[x_i, x_j]_{(e)}}$ as its ij^{th} entry is called GCED reciprocal LCEM matrix on T, where $(x_i, x_j)_{(e)}$ is the greatest common exponential divisor of x_i and x_j and $[x_i, x_j]_{(e)}$ is the greatest common exponential multiple of x_i and x_j.*

It is easy to see that GCED reciprocal LCEM matrices are symmetric. We always assume that $x_1 < x_2 < x_3 < \cdots < x_n$ in T. We define an arithmetic function $g(n)$ as follows:

$$g(n) = \sum_{a_1 b_1 = c_1} \cdots \sum_{a_r b_r = c_r} {p_1}^{2a_1} \cdots {p_r}^{2a_r} \mu^{(e)}({p_1}^{b_1} \cdots {p_r}^{b_r}) \tag{1}$$

where $n = {p_1}^{c_1} \cdots {p_r}^{c_r}$.

Theorem 2. *Let $R = \{y_1, y_2, \ldots, y_m\}$ be an exponential closure of the set $T = \{x_1, x_2, \ldots, x_n\}$, where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$.*

Define the $n \times m$ matrix $C = (c_{ij})$ by

$$c_{ij} = \begin{cases} \frac{1}{x_i}, & y_j |_e x_i \\ 0, & \text{otherwise} \end{cases}$$

and the $m \times m$ diagonal matrix by

$$\Psi = diag(g(x_1), g(x_2), \ldots, g(x_m)).$$

Then,

$$(T)_{(e)} = C \Psi C^t.$$

Proof. The ij^{th} entry of $C\Psi C^t$ is equal to

$$\begin{aligned}(C\Psi C^t)_{ij} &= \sum_{k=1}^{n} c_{ik} g(y_k) c_{jk} = \sum_{y_k|_e x_i, y_k|_e x_j} \frac{1}{x_i}\frac{1}{x_j} g(y_k) \\ &= \sum_{y_k|_e (x_i,x_j)_{(e)}} \frac{1}{x_i}\frac{1}{x_j} g(y_k),\end{aligned}$$

where the function g is defined in Equation 1. By Möbius inversion formula, we have,

$$\sum_{d|_e n} g(d) = n^2.$$

Finally, we get,

$$(C\Psi C^t)_{ij} = \frac{1}{x_i x_j}(x_i, x_j)_{(e)}^2 = \frac{(x_i, x_j)_{(e)}}{[x_i, x_j]_{(e)}}.$$

This completes the proof. □

Theorem 3. *Let $R = \{y_1, y_2, \ldots, y_m\}$ be an exponential closure of the set $T = \{x_1, x_2, \ldots, x_n\}$, where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$. Then*

$$\det(T)_{(e)} = \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} (\det C_{(k_1,k_2,\ldots,k_n)})^2 g(x_{k_1}) g(x_{k_2}) \ldots g(x_{k_n}),$$

where $C_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of C consisting of the ${k_1}^{th}$, ${k_2}^{th}$, $\ldots$, ${k_n}^{th}$ columns of C.

Proof. By Theorem 2, we have, $(T)_{(e)} = (C\Psi^{\frac{1}{2}})(C\psi^{\frac{1}{2}})^t$. Thus we can write $E = C\Psi^{\frac{1}{2}}$ which leads us to $(T)_{(e)} = EE^t$ and by applying Cauchy-Binet formula, we get

$$\begin{aligned}\det(T)_{(e)} &= \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} \det E_{(k_1,k_2,\ldots,k_n)} \det {E^t}_{(k_1,k_2,\ldots,k_n)} \\ &= \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} (\det E_{(k_1,k_2,\ldots,k_n)})^2,\end{aligned}$$

where $E_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of E consisting of the ${k_1}^{th}$, ${k_2}^{th}$, $\ldots$, ${k_n}^{th}$ columns of E and

$$\det E_{(k_1,k_2,\ldots,k_n)} = \sqrt{g(x_{k_1}) g(x_{k_2}) \ldots g(x_{k_n})} \det C_{(k_1,k_2,\ldots,k_n)}.$$

Hence,

$$\det(T)_{(e)} = \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} (\det C_{(k_1,k_2,\ldots,k_n)})^2 g(x_{k_1}) g(x_{k_2}) \ldots g(x_{k_n}).$$

This completes the proof. □

Corollary 4. *Let $T = \{x_1, x_2, \ldots, x_n\}$ be a finite ordered set of distinct positive integers. If $T = R$, then the determinant of GCED reciprocal LCEM matrix $(T)_{(e)}$ defined on T is given as:*

$$\det(T)_{(e)} = \prod_{k=1}^{n} \frac{1}{x_k^2} g(x_k).$$

Proof. Note that C is a lower triangular matrix with diagonal $(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n})$. This implies that $\det C = \prod_{k=1}^{n} \frac{1}{x_k}$. The determinant of a diagonal matrix is equal to the product of its diagonal entries which leads us to the desired result. □

Corollary 5. *If $(T)_{(e)}$ is an $n \times n$ GCED reciprocal LCEM matrix on a set $T = \{x_1, x_2, \ldots, x_n\}$, then the trace is given as*

$$tr(T)_{(e)} = n.$$

Lemma 6. *Let $(T)_{(e)} = (t_{ij})_{(e)}$ be an $n \times n$ GCED reciprocal LCEM matrix defined on an exponential divisor closed set T. Consider $n \times n$ matrix $C = (c_{ij})$ as defined in Theorem 2. Then, the $n \times n$ matrix $W = (w_{ij})$ defined by*

$$w_{ij} = \begin{cases} x_j \mu^{(e)}(\frac{x_i}{x_j}), & x_j|_e x_i \\ 0, & \textit{otherwise} \end{cases}$$

is the inverse of the matrix C.

Proof. The ij^{th} entry of CW is given by

$$(CW)_{ij} = \sum_{k=1}^{n} c_{ik} w_{kj} = \sum_{x_k|_e x_i, x_j|_e x_k} \mu^{(e)}(\frac{x_j}{x_i})(\frac{x_j}{x_k})$$

$$= \frac{x_j}{x_i} \sum_{x_d|_e \frac{x_i}{x_j}} \mu^{(e)}(x_d) = \begin{cases} 1, & if \;\; x_i = x_j \\ 0, & \text{otherwise} \end{cases}$$

If $\frac{x_i}{x_j}$ is not an integer, then no x_d divides $\frac{x_i}{x_j}$. If $x_i = x_j$ then, $1|_e 1$ and $\mu^{(e)}(1) = 1$. □

Theorem 7. *Let $(T)_{(e)}$ be $n \times n$ GCED reciprocal LCEM matrix on an exponential divisor closed set. Then, its inverse matrix $(A)_{(e)} = (a_{ij})_{(e)}$ is given as:*

$$(a_{ij})_{(e)} = x_i x_j \sum_{x_i|_{(e)} x_d, x_j|_{(e)} x_d} \frac{\mu^{(e)}(\frac{x_d}{x_i})\mu^{(e)}(\frac{x_d}{x_j})}{g(x_d)}.$$

Proof. Since $(T)_{(e)} = (C\Psi C^t)$ and by Lemma 6, we have $C^{-1} = W$, then

$$(T)_{(e)}^{-1} = (C\Psi C^t)^{-1} = W^t \Psi^{-1} W,$$

where ij^{th} entry of $(T)_{(e)}^{-1}$ is given as

$$(a_{ij})_{(e)} = x_i x_j \sum_{x_i|_{(e)} x_d, x_j|_{(e)} x_d} \frac{\mu^{(e)}(\frac{x_d}{x_i})\mu^{(e)}(\frac{x_d}{x_j})}{g(x_d)}$$

which completes the proof. □

3 Numerical results

Example 8. *Let $T = \{12, 18, 36\}$. The GCED reciprocal LCEM matrix $(T)_{(e)}$ on T is given as:*

$$(T)_{(e)} = \begin{bmatrix} 1 & \frac{6}{36} & \frac{12}{36} \\ \frac{6}{36} & 1 & \frac{18}{36} \\ \frac{12}{36} & \frac{18}{36} & 1 \end{bmatrix}$$

Note that $A = \{12, 18, 36\}$ is not an exponential divisor closed set. Its exponential closure is $R = \{6, 12, 18, 36\}$. The 3×4 matrix $(C)_{(e)}$ is

$$C = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & 0 & 0 \\ \frac{1}{18} & 0 & \frac{1}{18} & 0 \\ \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \end{bmatrix}$$

By Theorem 3, we know that,

$$\det(T)_{(e)} = \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} (\det C_{k_1, k_2, \ldots, k_n})^2 g(x_{k_1}) g(x_{k_2}) \ldots g(x_{k_n}).$$

So,

$$\det(T)_{(e)} = \begin{vmatrix} \frac{1}{12} & \frac{1}{12} & 0 \\ \frac{1}{18} & 0 & \frac{1}{18} \\ \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \end{vmatrix}^2 g(6)g(12)g(18) + \begin{vmatrix} \frac{1}{12} & \frac{1}{12} & 0 \\ \frac{1}{18} & 0 & 0 \\ \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \end{vmatrix}^2 g(6)g(12)g(36) +$$

$$\begin{vmatrix} \frac{1}{12} & 0 & 0 \\ \frac{1}{18} & \frac{1}{18} & 0 \\ \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \end{vmatrix}^2 g(6)g(18)g(36) + \begin{vmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{18} & 0 \\ \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \end{vmatrix}^2 g(12)g(18)g(36),$$

where $g(6) = 36, g(12) = 108, g(18) = 288, g(36) = 864$.
Hence, the determinant is given as: $\det(T)_{(e)} = \frac{1}{(7776)^2}[(36)(108)(288) + (36)(108)(864) + (36)(288)(864) + (108)(288)(864)] = 0.6667$

Example 9. *Let $T = \{2, 4, 16\}$. The set is exponential divisor closed, so we apply the corollary to Theorem 3 directly to calculate the determinant. The GCED reciprocal LCEM defined on T is given as:*

$$(T)_{(e)} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix}$$

$g(2) = 2^2\mu^{(e)}(2) = 4, g(4) = (2)^2\mu^{(e)}(2^2) + (4)^2\mu^{(2)}(2) = 12,\ g(16) = (2)^2\mu^{(e)}(2^4) + (4)^2\mu^{(e)}(2^2) + (16)^2\mu^{(e)}(2) = 240$. *Then,*

$$\det(T)_{(e)} = \prod_{k=1}^{3} \frac{1}{x_k^2} g(x_k) = \frac{(4)(12)(240)}{(4)(16)(256)} = 0.703125.$$

Example 10. *Let $T = \{2, 4, 16\}$. The 3×3 GCED reciprocal LCEM matrix $(T)_{(e)}$ defined on T is given as:*

$$(T)_{(e)} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix}$$

By Theorem 7, we know that $(T)^{-1}{}_{(e)} = (a_{ij})$ where,

$$a_{11} = 4\Big[\sum_{2|_e x_k} \frac{\mu^{(e)}(\frac{x_k}{2})\mu^{(e)}(\frac{x_k}{2})}{g(x_k)}\Big] = 4\Big[\frac{\mu^{(e)}(2)\mu^{(e)}(2)}{g(2)} + \frac{\mu^{(e)}(2^2)\mu^{(e)}(2^2)}{g(4)} + \frac{\mu^{(e)}(2^4)\mu^{(e)}(2^4)}{g(16)}\Big] = \frac{4}{3},$$
$$a_{12} = 8\Big[\frac{\mu^{(e)}(2^2)\mu^{(e)}(2)}{g(4)} + \frac{\mu^{(e)}(2^4)\mu^{(e)}(2^2)}{g(16)}\Big] = \frac{-2}{3},\ a_{13} = 32\Big[\frac{\mu^{(e)}(2^4)\mu^{(e)}(2)}{g(16)}\Big] = 0.$$

Similarly, one can calculate and verify the following values:

$$a_{22} = \frac{7}{5}, a_{23} = \frac{-4}{15}, a_{33} = \frac{16}{15}.$$

So, the inverse of the GCED reciprocal LCEM matrix $(T)_{(e)}$ is given as:

$$(T)^{-1}{}_{(e)} = \begin{bmatrix} \frac{4}{3} & \frac{-2}{3} & 0 \\ \frac{-2}{3} & \frac{7}{5} & \frac{-4}{15} \\ 0 & \frac{-4}{15} & \frac{16}{15} \end{bmatrix}.$$

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