A note on generalized Tribonacci sequence

Aldous Cesar F. Bueno

Department of Mathematics and Physics, Central Luzon State University
Science City of Muñoz 3120, Nueva Ecija, Philippines

Abstract: In this study, we provide a property of the generalized Tribonacci sequence through limits.

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1 Introduction

The Tribonacci sequence $\{T_k\}_{k=0}^{+\infty}$ satisfy the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

having the initial values $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$. Furthermore, its Binet’s formula is given by

$$T_n = \frac{\tau^{n+1}}{(\tau - \sigma)(\tau - \rho)} + \frac{\sigma^{n+1}}{(\sigma - \tau)(\sigma - \rho)} + \frac{\rho^{n+1}}{(\rho - \tau)(\rho - \sigma)},$$

where $\tau$, $\sigma$ and $\rho$ are the roots of $x^3 - x^2 - x - 1 = 0$. These roots have the following properties:

$$\lim_{n \to +\infty} \frac{\sigma^n}{\tau^n} = 0,$$

$$\lim_{n \to +\infty} \frac{\rho^n}{\tau^n} = 0,$$

and due to these we have,

$$\lim_{n \to +\infty} \frac{T_{n+1}}{T_n} = \tau.$$

Actually, the root $\tau$ is called the Tribonacci constant and its explicit form is

$$\tau = 1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}.$$
On the other hand, the generalized Tribonacci sequence denoted by \( \{ S_k \}_{k=0}^{+\infty} \) satisfy the recurrence relation
\[
S_n = S_{n-1} + S_{n-2} + S_{n-3},
\]
where the initial values \( S_1, S_2 \) and \( S_3 \) are arbitrary but are not simultaneously zero.

Natividad and Policarpio [1] provided a formula for finding its \( n \)-th term and it is given by
\[
S_n = T_{n-2} S_1 + (T_{n-2} + T_{n-3}) S_2 + T_{n-1} S_3.
\]

Our goal in this study is to investigate the generalized Tribonacci sequence through limits specifically, where we will be dealing on the limit given by
\[
\lim_{n \to +\infty} \frac{S_{n+j}}{S_n},
\]
where \( j \) is a positive integer.

## 2 Main results

**Theorem.**
\[
\lim_{n \to +\infty} \frac{S_{n+j}}{S_n} = \tau^j.
\]

**Proof:**
\[
\lim_{n \to +\infty} \frac{S_{n+j}}{S_n} = \lim_{n \to +\infty} \frac{T_{n+j-2} S_1 + (T_{n+j-2} + T_{n+j-3}) S_2 + T_{n+j-1} S_3}{T_{n-2} S_1 + (T_{n-2} + T_{n-3}) S_2 + T_{n-1} S_3}
\]
\[
= \lim_{n \to +\infty} \frac{T_{n+j-2} S_1 / T_n + (T_{n+j-2} + T_{n+j-3}) S_2 / T_n + T_{n+j-1} S_3 / T_n}{T_{n-2} S_1 / T_n + (T_{n-2} + T_{n-3}) S_2 / T_n + T_{n-1} S_3 / T_n}.
\]

**Claim:**
\[
\lim_{n \to +\infty} \frac{T_{n+j}}{T_n} = \tau^j
\]
\[
= \lim_{n \to +\infty} \frac{\tau^{n+j+1}}{(\tau-\sigma)(\tau-\rho)} \left( \frac{\sigma^{n+1}}{(\sigma-\tau)(\sigma-\rho)} + \frac{\rho^{n+1}}{(\rho-\tau)(\rho-\sigma)} \right)
\]
\[
= \frac{\tau^{j+1}}{\tau^j}; \text{ by the properties of the roots } \tau, \sigma, \text{ and } \rho
\]
\[
= \tau^j.
\]
Continuing and using our proven claim, we obtain

\[
\begin{align*}
\tau^{j-2}S_1 + (\tau^{j-2} + \tau^{j-3})S_2 + \tau^{j-1}S_3 &= \tau^{-2}S_1 + (\tau^{-2} + \tau^{-3})S_2 + \tau^{-1}S_3 \\
\tau^{j+1}S_1 + (\tau^{j+1} + \tau^j)S_2 + \tau^{j+2}S_3 &= \tau S_1 + (\tau + 1)S_2 + \tau^2S_3 \\
\tau S_1 + (\tau + 1)S_2 + \tau^2S_3 &= \tau^j,
\end{align*}
\]

as desired. \(\square\)

**Corollary.**

\[
\lim_{n \to +\infty} \frac{S_{n+1}}{S_n} = \tau.
\]

### 3 Conclusion

In summary, we have established that like the Tribonacci sequence, the ratio of the two terms \(S_{n+j}\) and \(S_n\) of the generalized Tribonacci sequence, given by \(\frac{S_{n+j}}{S_n}\), approaches the value \(\tau^j\) as \(n\) tends to infinity where \(\tau\) is the Tribonacci constant.

### References
