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An infinite primality conjecture for prime-subscripted Fibonacci numbers

J. V. Leyendekkers[1] and A. G. Shannon[2,3]

[1] Faculty of Science, The University of Sydney
NSW 2006, Australia

[2] Faculty of Engineering & IT, University of Technology
Sydney, NSW 2007, Australia
[3] Campion College
PO Box 3052, Toongabbie East, NSW 2146, Australia
e-mails: t.shannon@warrane.unsw.edu.au,
Anthony.Shannon@uts.edu.au

Abstract: The row structures of the prime-subscripted Fibonacci numbers in the modular ring Z_4 show distinction between primes and composites. The class structure of the Fibonacci numbers suggest that these row structures must survive to infinity and hence that Fibonacci primes must too. The functions $F_p = Kp \pm 1$ and F_p (factors) $= kp \pm 1$ support the structural evidence. The graph of (K/k) versus p displays a Raman-spectra form persisting to infinity: $\ln(K/k)$ is linear in p in the composite case while primes lie along the p-axis to infinity.
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1 Introduction

On knowing the infinite, Franklin has this to say: "It is evident that the idea of an infinite structure cannot be derived purely from perceptual experience [...] Our perceptual experience is finite in character" [2]. This paper will explore an aspect of infinity in the context of Fibonacci primes and the regularity of the Fibonacci numbers generated from the second order homogeneous recurrence relation

$$F_n = F_{n-1} + F_{n-2},\ n > 2, \tag{1.1}$$

which gives rise to the many periodicities found in this sequence [14] and its very precise integer structure; for example, in the associated modular ring Z_4 (Table 1).

Row $r_i \downarrow$	**Class** $i \rightarrow$	$\overline{0}_4$	$\overline{1}_4$	$\overline{2}_4$	$\overline{3}_4$	**Comments**
0		0	1	2	3	• $N = 4r_i + i$
1		4	5	6	7	• even $\overline{0}_4$, 2_4
2		8	9	10	11	• $\left(N^n, N^{2n}\right) \in \overline{0}_4$
3		12	13	14	15	• odd $\overline{1}_4, \overline{3}_4$; $N^{2n} \in \overline{1}_4$

Table 1. Classes and rows for Z_4

This particular ring is the most appropriate in this context because the prime-subscripted Fibonacci numbers satisfy [1, 15]

$$F_p = F^2_{\frac{p+1}{2}} + F^2_{\frac{p-1}{2}} \tag{1.2}$$

and the only odd class in Z_4, which can form this sum of squares is $\overline{1}_4$, which is generated by $4r_1 + 1$ [12]; that is, prime-subscripted Fibonacci numbers will always fall in this class. When F_p is itself prime, Equation (1.2) is the only possible sum [12], but not with composites [7, 8].

2 Class structures with Fibonacci numbers

The prime-subscripted Fibonacci numbers will now be considered in detail in order to assess the evidence of infinitely many Fibonacci primes. Since there are infinitely many primes, F_p will have infinitely many values, but since F_p may be composite, the number of prime-subscripted Fibonacci primes may be finite. The class structure for the Fibonacci numbers in the modular ring Z_4 is.

$$\overline{1}_4\overline{1}_4\overline{2}_4\overline{3}_4\overline{1}_4\overline{0}_4\overline{1}_4\overline{1}_4\overline{2}_4\overline{3}_4\overline{1}_4\overline{0}_4\overline{1}_4\overline{1}_4\overline{2}_4\overline{3}_4\overline{1}_4\overline{0}_4 \ldots \tag{2.1}$$

which is repeated to infinity as the formation of a recursive sequence does not change [5]. Since $F_p \in \overline{1}_4$, the row structure is given by

$$F_p = 4r_1 + 1. \tag{2.2}$$

The row structures of F_p for $p = 7$ to 101 are displayed in Table 2 according to p^*, the right-end-digit of p (that is, p (mod 10)). This p^* defines the class of p in the modular ring Z_5 [8, 12, 18].

As can be seen there is a distinction between primes and composites which is worth exploring further. If there are no primes for very large p, then the row structure of F_p would be very restricted for $p^* = 1$ or 3, and some row structures for $p^* = 7$ or 9 would not occur. In view of the precise nature of the F_p structure this would not be possible. This makes a compelling case for infinitely many prime Fibonacci numbers.

p*	Primes	Composites
1	$\bar{1}_4\bar{0}_4\bar{1}_4$, $\bar{1}_4\bar{1}_4\bar{0}_4$, $\bar{1}_4\bar{2}_4\bar{0}_4$, $\bar{1}_2\bar{2}_4\bar{1}_4$	$\bar{1}_4\bar{3}_4\bar{1}_4$
3	$\bar{1}_4\bar{0}_4\bar{3}_4$, $\bar{1}_4\bar{1}_4\bar{2}_4$, $\bar{1}_4\bar{2}_4\bar{2}_4$	$\bar{1}_4\bar{1}_4\bar{3}_4$
7	$\bar{1}_4\bar{0}_4\bar{2}_4$, $\bar{1}_4\bar{3}_4\bar{3}_4$	$\bar{1}_4\bar{0}_4\bar{3}_4$, $\bar{1}_4\bar{1}_4\bar{1}_4$, $\bar{1}_4\bar{2}_4\bar{3}_4$
9	$\bar{1}_4\bar{1}_4\bar{1}_4$	$\bar{1}_4\bar{1}_4\bar{0}_4$, $\bar{1}_4\bar{1}_4\bar{2}_4$, $\bar{1}_4\bar{3}_4\bar{0}_4$, $\bar{1}_4\bar{3}_4\bar{3}_4$

Table 2. Row structures of F_p

3 Functions and factors

Here we consider functions and factors [10, 11] defined respectively by $F_p = pK \pm 1$ and F_p (factors) $= pk \pm 1$. When K/k for p 7 to 101 is plotted as function of p, the result is a Raman-like spectra [13] with a base of unity (since $K = k$ for primes), and a variety of bands when $K/k \neq 1$ (representing the composites). Continuous bands without the base of 1 would have to occur if no more primes occur for large p. This would be inconsistent with any normal spectra and would indicate severe rupture of the F_p-sequence structure which would not be possible in view of the formation mechanism of the Fibonacci recurrence relation.

ln(K/k) as a function of p: for composite F_p this function is linear and passes through the origin where $K = k$ and $\ln(K/k) = 0$; that is, for composites only

$$\ln(K/k) \approx \frac{p}{10(\varphi - 1)} \tag{3.1}$$

in which φ is the Golden Ratio [14]. In Table 3 the standardized values represented by $\ln(K/k)^{\ddagger}$ and $p/10(\varphi\text{-}1)^{\ddagger}$, respectively are set out to show that the two sides of (3.1) are proportionally approximate. The following standard normalization formula was utilized [6]

$$x_i^{\ddagger} = \frac{x_i - x_{\min}}{x_{\max} - x_{\min}} \tag{3.2}$$

p	$\ln(K/k)^{\ddagger}$	$p/10(\varphi\text{-}1)^{\ddagger}$
19	0	0
31	0.1	0.1
37	0.2	0.1
41	0.2	0.3
53	0.4	0.4
59	0.5	0.5
67	0.6	0.6
79	0.8	0.8
89	0.9	0.9
97	1	1

Table 3. Composite F_p

When F_p is prime the line coincides with the p-axis which meets the composite line at the origin and increases asymptotically.

Another parameter which supports this structural evidence is S_p [3]:

$$S_p = \sum_{i=1}^{p} F_i = F_{p+2} - 1 \tag{3.3}$$

from which, S_p^*, the right-end-digits for S_p, for primes are distinct from those of composite F_p (Table 4).

p*	**S_p^***	
	Primes	**Composites**
1	3	6, 8
3	5, 9	4
7	1, 4, 8	5
9	9	6

Table 4. S_p, $3 \leq p \leq 97$

The stability of S_p* is based on the structural ability of the Fibonacci numbers from their very definition by means of a linear recurrence relation (1.1) and the periodicity of their class structure in the modular ring (2.1).

4 Concluding comments

This class pattern of F_p is invariant since the mechanism of generation of the Fibonacci numbers remains the same to infinity. If class structure is invariant, then the demonstrated difference between the row structures for primes and composites should also be invariant. The relationships of K and k with p also show that primes are generated as long as primes exist; that is, since there is an infinity of primes [19], then there must also be an infinity of prime-subscripted Fibonacci prime numbers.

Another possible line of approach for further research related to the central issue in this paper would be by means of asymptotic proofs; that is, for 'almost all n'. These have been used previously for Fibonacci numbers by Horadam and Subba Rao [4, 16, 17].

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