A basic logarithmic inequality, 
and the logarithmic mean

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Abstract: By using the basic logarithmic inequality \( \ln x \leq x - 1 \) we deduce integral inequalities, which particularly imply the inequalities \( G < L < A \) for the geometric, logarithmic, resp. arithmetic means.

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1 Introduction

Let \( a, b > 0 \). The logarithmic mean \( L = L(a, b) \) of \( a \) and \( b \) is defined by

\[
L = L(a, b) = \frac{b - a}{\ln b - \ln a} \quad \text{for} \quad a \neq b \quad \text{and} \quad L(a, a) = a. 
\]

(1)

Let \( G = G(a, b) = \sqrt{ab} \) and \( A = A(a, b) = \frac{a + b}{2} \) denote the classical geometric, resp. logarithmic means of \( a \) and \( b \).

One of the most important inequalities for the logarithmic mean (besides e.g. \( a < L(a, b) < b \) for \( a < b \)) is the following:

\[
G < L < A \quad \text{for} \quad a \neq b 
\]

(2)

The left side of (2) was discovered by B. C. Carlson in 1966 ([1] see [2]), while the right side in 1957 by B. Ostle and H. L. Terwilliger [3].

We note that relation (2) has applications in many subject of pure or applied mathematics and physics including e.g. electrostatics, probability and statistics, etc. (see e.g. [4, 5]).

The following basic logarithmic inequality is well-known:
Theorem 1.

\[
\ln x \leq x - 1 \text{ for all } x > 0. \tag{3}
\]

There is equality only for \( x = 1 \).

Inequality (3) may be proved e.g. by considering the auxiliary function

\[ f(x) = x - \ln x - 1, \]

and it is easy to show that \( x = 1 \) is a global minimum to \( f \), so

\[ f(x) \geq f(1) = 0. \]

Another proof is based on the Taylor expansion of the exponential function, yielding

\[ e^t = 1 + t + \frac{t^2}{2} \cdot e^\theta, \text{ where } \theta \in (0, t). \]

Put \( t = x - 1 \), and (3) follows.

The continuous arithmetic, geometric and harmonic means of positive, integrable function \( f : [a, b] \to \mathbb{R} \) are defined by

\[
A_f = \frac{1}{b-a} \int_a^b f(x)dx, \quad G_f = e^{\frac{1}{b-a} \int_a^b \ln f(x)dx}
\]

and

\[
H_f = \frac{b-a}{\int_a^b dx/f(x)},
\]

where \( a < b \) are real numbers.

By using (3) we will prove the following classical fact:

Theorem 2.

\[
H_f \leq G_f \leq A_f \tag{4}
\]

Then, by applying (4) for certain particular functions, we will deduce (2). In fact, (2) will be obtained in a stronger form. The main idea of this note is the use of very simple inequality (3) in the theory of means.

2 The proofs

Proof of Theorem 2. Put

\[
x = \frac{(b-a)f(t)}{\int_a^b f(t)dt}
\]

in (3), and integrate on \( t \in [a, b] \) the obtained inequality. One gets

\[
\int_a^b \ln f(t)dt - \left( \frac{1}{b-a} \int_a^b f(t)dt \right) (b-a) \leq \frac{(b-a) \int_a^b f(t)dt}{\int_a^b f(t)dt} - (b-a) = 0.
\]
This gives the right side of (4).
Apply now this inequality to \( \frac{1}{f} \) in place of \( f \). As
\[
\ln \frac{1}{f(t)} = -\ln f(t),
\]
we immediately obtain the left side of (4).

**Corollary 1.** If \( f \) is as above, then
\[
\left( \int_a^b f(t)\,dt \right) \left( \int_a^b \frac{1}{f(t)}\,dt \right) \geq (b - a)^2. \tag{5}
\]
This follows by \( H_f \leq A_f \) in (4).

**Remark 1.** Let \( f \) be continuous in \([a, b]\). The above proof shows that there is equality e.g. in right side of (4) if
\[
f(t) = \frac{1}{b - a} \int_a^b f(t)\,dt. \tag{6}
\]
By the first mean value theorem of integrals, there exists \( c \in [a, b] \) such that
\[
\frac{1}{b - a} \int_a^b f(t)\,dt = f(c).
\]
Since by (6) one has \( f(t) = f(c) \) for all \( t \in [a, b] \), \( f \) is a constant function.
When \( f \) is integrable, as
\[
\int_a^b \ln \left( b - a \frac{f(t)}{\int_a^b f(t)\,dt} \right) \,dt = 0,
\]
as for \( g(t) = \ln \frac{(b - a)f(t)}{\int_a^b f(t)\,dt} > 0 \) one has
\[
\int_a^b g(t)\,dt = 0,
\]
it follows by a known result that \( g(t) = 0 \) almost everywhere (a.e.). Therefore
\[
f(t) = \frac{1}{b - a} \int_a^b f(t)\,dt
\]
a.e., thus \( f \) is a constant a.e.

**Remark 2.** If \( f \) is continuous, it follows in the same manner, that in the left side of (4) there is equality only for \( f = \text{constant} \). The same is true for inequality (5).
Proof of (2). Apply $G_f \leq A_f$ to $f(x) = \frac{1}{x}$. Remark that
\[
\frac{1}{b-a} \int_a^b \ln x \, dx = \ln I(a,b),
\]
where $a < I(a,b) < b$.

This mean is known in the literature as "identric mean" (see e.g. [4]). As $f(x) = \frac{1}{x}$ is not constant, we get by
\[
A_f = \frac{1}{L(a,b)}, \quad G_f = \frac{1}{I(a,b)},
\]
that
\[
L < I. \tag{7}
\]

Applying the same inequality $G_f \leq A_f$ to $f(x) = x$ one obtains
\[
I < A. \tag{8}
\]

Remark 3. Inequalities (7) and (8) can be deduced at once by applying all relations of (4) to $f(x) = x$. Apply now (5) to $f(t) = e^t$. After elementary computations, we get
\[
\frac{e^b - e^a}{b-a} > e^{\frac{a+b}{2}}. \tag{9}
\]

As $f(t) > 0$ for any $t \in \mathbb{R}$, inequality (9) holds true for any $a, b \in \mathbb{R}, b > a$. Replace now $b := \ln b, a := \ln a$, where now the new values of $a$ and $b$ are $> 0$. One gets from (9):
\[
L > G. \tag{10}
\]

By taking into account of (7) -- (10), we can write:
\[
G < L < I < A, \tag{11}
\]
i.e. (2) is proved (in improved form on the right side).

Remark 4. Inequality (4) (thus, relation (10)) follows also by $G_f \leq A_f$ applied to $f(t) = e^t$.

Remark 5. The right side of (2) follows also from (5) by the application $f(t) = t$. As
\[
\int_a^b t \, dt = \frac{b^2 - a^2}{2} \quad \text{and} \quad \int_a^b \frac{1}{t} \, dt = (\ln b - \ln a),
\]
the relation follows.

Remark 6. Clearly, in the same manner as (4), the discrete inequality of means can be proved, by letting $x = \frac{nx_1}{x_1 + \ldots + x_n} (x_1, \ldots, x_n > 0)$. 

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References


