A note on the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted \( p \)-adic invariant integral on \( \mathbb{Z}_p \)

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Abstract: In this paper we will give the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted \( p \)-adic \( q \)-measure on \( \mathbb{Z}_p \). In special case, if there is no twisted, then we can derive the same result as Jeong and Rim, 2012; If the case weight zero and no twist, then we derive the same result as Kim 2012.

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1 Introduction

Let \( p \) be a fixed odd prime number. Throughout this paper, the symbols \( \mathbb{Z}_p \), \( \mathbb{Q}_p \), and \( \mathbb{C}_p \) denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of the algebraic closure of \( \mathbb{Q}_p \), respectively. The \( p \)-adic norm \( |.|_p \) is defined by \( |x|_p = p^{-v_p(x)} = p^{-r} \) for \( x = p^rs^t \), where \( s \) and \( t \) are integers with \( (p,s) = (p,t) = 1 \) and \( r \in \mathbb{Q} \) (see [1–16]).

When one speaks of \( q \)-extension, \( q \) can be regard as an indeterminate, a complex \( q \in \mathbb{C} \), or a \( p \)-adic number \( q \in \mathbb{C}_p \). In this paper we assume that \( q \in \mathbb{C}_p \) with \( |q - 1|_p < p^{-\frac{1}{p^\infty}} \) and we use the notations of \( q \)-numbers as follows:

\[
[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q},
\]
For \( n \in \mathbb{Z}_+ \), let \( C_{p^n} = \{ w \mid w^{p^n} = 1 \} \) be the cyclic group of order \( p^n \) and let \( T_p \) be the space of locally constant space, i.e., \( T_p = \lim_{n \to \infty} C_{p^n} = \cup_{n \geq 0} C_{p^n} \).

For any positive integer \( N \), let

\[
a + p^N \mathbb{Z}_p = \{ x \in \mathbb{Z}_p \mid x \equiv a \pmod{p^N} \}
\]

(1)

where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < p^N \)(see [1, 2, 3, 5-9]).

It is known that the fermionic \( p \)-adic \( q \)-measure on \( \mathbb{Z}_p \) is given by Kim as follows:

\[
\mu_{-q}(a + p^N \mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}} = \frac{1 + q + q^{p^n} (-q)^a}{1 + q^{p^N} (-q)}
\]

(2)

(see [5, 7, 12–16]).

Let \( C(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \). From (2), the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x,
\]

(3)

where \( f \in C(\mathbb{Z}_p) \)(see [1, 5, 7, 12-16]).

For \( w \in T_p \), we consider the twisted \( q \)-Euler polynomials \( \varepsilon_{n,q}^w(x) \) as

\[
\int_{\mathbb{Z}_p} q^{-y} w^y e^{[x+y]q} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \varepsilon_{n,q}^w(x) \frac{t^n}{n!}
\]

(4)

From (4), we can derive the following equation.

\[
\varepsilon_{n,q}^w(x) = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} (-1)^l \frac{q^lx}{1 + wq^l},
\]

(5)

(see [16]).

From (5), we note that \( \lim_{q \to 1} \varepsilon_{n,q}^w(x) = \varepsilon_{n,q}^w(x) \).

In special case, \( x = 0 \), we have \( \varepsilon_{0,q}^w(0) = \varepsilon_{0,q}^w \) are the \( n \)-th twisted \( q \)-Euler number, we have

\[
\varepsilon_{n,q}^w = \int_{\mathbb{Z}_p} q^{-y} w^y [x]_q^n d\mu_{-q}(y) = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} (-1)^l \frac{1}{1 + wq^l}.
\]

(6)

In special, we have

\[
\varepsilon_{n,q}^w = \frac{[2]_q}{1 + w}.
\]

(7)

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) fermionic measure \( \mathbb{Z}_p \) satisfying

\[
|\mu_{-1}(a + p^n \mathbb{Z}_p) - \mu_{-1}(a + p^{n+1} \mathbb{Z}_p)|_p \leq \delta_{n,q},
\]

(8)

where \( \delta_{n,q} \to 0 \), \( a \) is an element of \( \mathbb{Z}_p \), and \( \delta_{n,q} \) is independent of \( a \) (for strongly fermionic measure, \( \delta_{n,q} \) is replaced by \( Cp^{-n} \), where \( C \) is a positive constant)(see [5, 7, 8, 13]).
Let \( f(x) \) be a function defined on \( \mathbb{Z}_p \). The fermionic integral of \( f \) with respect to a weakly fermionic measure \( \mu_{-1} \) is

\[
\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \to \infty} \sum_{x=0}^{p^n-1} f(x) \mu_{-1}(x + p^n \mathbb{Z}_p),
\]

if the limit exists.

If \( \mu_{-1} \) is a weakly fermionic measure on \( \mathbb{Z}_p \), then the Radon–Nikodym derivative of \( \mu_{-1} \) with respect to the Haar measure on \( \mathbb{Z}_p \) as follows:

\[
f_{\mu_{-1}}(x) = \lim_{n \to \infty} \mu_{-1}(x + p^n \mathbb{Z}_p),
\]

(see [5, 7]).

Note that \( f_{\mu_{-1}} \) is only a continuous function on \( \mathbb{Z}_p \). Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), let us define \( \mu_{-1,f} \) as follows:

\[
\mu_{-1,f}(x + p^n \mathbb{Z}_p) = \int_{x+p^n \mathbb{Z}_p} f(x) d\mu_{-1}(x),
\]

(10)

where the integral is the fermionic \( p \)-adic invariant integral. From (9), we can easily note that \( \mu_{-1,f} \) is a strongly fermionic measure on \( \mathbb{Z}_p \)(see [5, 7]). Since

\[
\left| \mu_{-1,f}(x + p^n \mathbb{Z}_p) - \mu_{-1,f}(x + p^{n+1} \mathbb{Z}_p) \right|_p = \left| \sum_{x=0}^{p^n-1} f(x)(-1)^x - \sum_{x=0}^{p^n} f(x)(-1)^x \right|_p = \left| \frac{f(p^n)}{p^n} \right|_p \leq C p^{-n},
\]

where \( C \) is positive constant.

In this paper, we will give the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted \( p \)-adic \( q \)-measure on \( \mathbb{Z}_p \). In special case, if there is no twisted, then we can derive the same result as Jeong and Rim, 2012 (see [5]). In the case of weight zero and no twisted, then we derive the same result as Kim, 2012 (see [7]).

2 The Lebesgue–Radon–Nikodym theorem with respect to the weighted \( p \)-adic \( q \)-measure

For any positive integer \( a \) and \( n \) with \( a < p^n \) and \( f \in UD(\mathbb{Z}_p) \), we define \( \tilde{\mu}_{f, -q}^w \), weighted and twisted fermionic measure on \( \mathbb{Z}_p \) as follows:

\[
\tilde{\mu}_{f, -q}^w(a + p^n \mathbb{Z}_p) = \int_{a+p^n \mathbb{Z}_p} w^x q^x f(x) d\mu_{-1}(x)
\]

(11)

where the integral is the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \).

From (11), we note that
\[
\tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) = \lim_{m \to \infty} \sum_{x=0}^{p^n-1} f(a + p^n x) (-1)^{a + p^n x} q^{a + p^n x} w^{a + p^n x} \\
= (-1)^a w^a q^a \lim_{m \to \infty} \sum_{x=0}^{p^{m-n}-1} f(a + p^n x) (-1)^x q^{p^n x} \\
= (-1)^a \int_{\mathbb{Z}_p} w^a f(a + p^n x) q^{a + p^n x} d\mu_{-1}(x). 
\] (12)

By (12), we get

\[
\tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) = (-1)^a \int_{\mathbb{Z}_p} w^a f(a + p^n x) q^{a + p^n x} d\mu_{-1}(x). 
\] (13)

Thus, by (13), we have

\[
\tilde{\mu}_{\alpha f + \beta g,-q} = \alpha \tilde{\mu}_{f,-q} + \beta \tilde{\mu}_{g,-q}, 
\] (14)

where \(f, g \in UD(\mathbb{Z}_p)\) and \(\alpha, \beta\) are positive constants.

By (11), (12), (13) and (14), we get

\[
\|\tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p)\|_p \leq \|f_{wq}\|_\infty, 
\] (15)

where \(\|f_{wq}\|_\infty = \sup_{x \in \mathbb{Z}_p} |f(x)q^x w^x|_p\).

Let \(P(x) \in \mathbb{C}_p[[x]]\) be an arbitrary \(q\)-polynomial. Now we show \(\tilde{\mu}_{P,-q}\) is a strongly weighted and twisted fermionic \(p\)-adic invariant measure on \(\mathbb{Z}_p\). Without a loss of generality, it is enough to prove the statement for \(P(x) = [x]^k_q\).

For \(a \in \mathbb{Z}\) with \(0 \leq a < p^n\), we have

\[
\tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p) = \lim_{m \to \infty} (-q)^a w^a \sum_{i=0}^{p^{m-n}-1} q^{p^n i} (a + ip^n)_q (-1)^i. 
\] (16)

Note that

\[
[a + ip^n)_q = ([a]_q + q^a [p^n]_q [i]_q w^n)^k. 
\] (17)

By (6), (16) and (17),

\[
\tilde{\mu}_{P,-q} = \left\{ [a]_q^k [z]_0, w^n + k[a]_q^{k-1} q^a [p^n]_q [i]_q w^n + \cdots + q^k [p^n]_q [i]_q w^n \right\}. 
\]

By (7), (12) and (13), we easily get

\[
\tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p) \equiv (-1)^a q^a_w [a]_q^k [z]_0, w^n \pmod{[p^n]_q} \\
\equiv (-1)^a 1 + q^a_w P(a) q^a_w \pmod{[p^n]_q}. 
\] (18)

For \(x \in \mathbb{Z}_p\), let \(x \equiv x_n \pmod{p^n}\) and \(x \equiv x_{n+1} \pmod{p^{n+1}}\), where \(x_n, x_{n+1} \in \mathbb{Z}\) with \(0 \leq x_n < p^n\) and \(0 \leq x_{n+1} < p^{n+1}\). Then we have

\[
\|\tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p) - \tilde{\mu}_{P,-q}(a + p^{n+1} \mathbb{Z}_p)\|_p \leq C p^{-\nu_p(1-q^p^n)}, 
\] (19)
where $C$ is a positive constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}_{P,-q}}(a) = \lim_{n \to \infty} \tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p) = (-1)^a q^a w^a.$$  

Then, by (18) and (19), we see that

$$f_{\tilde{\mu}_{P,-q}}(a) = (-1)^a q^a w^a [a]_q! = (-1)^a q^a w^a P(a).$$  \hspace{1cm} (20)

Since $f_{\tilde{\mu}_{P,-q}}(x)$ is a continuous function on $\mathbb{Z}_p$, for $x \in \mathbb{Z}_p$, we have

$$f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x w^x [x]_q!^k, (k \in \mathbb{Z}_+).$$ \hspace{1cm} (21)

Let $g \in UD(\mathbb{Z}_p)$. Then, by (19), (20) and (21), we get

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) = \lim_{n \to \infty} p^n - 1 \sum_{x=0}^{p^n-1} g(x) \tilde{\mu}_{P,-q}(x + p^n \mathbb{Z}_p)$$

$$= \lim_{n \to \infty} p^n - 1 \sum_{x=0}^{p^n-1} g(x) q^x w^x [x]_q^k (-1)^x$$

$$= \int_{\mathbb{Z}_p} g(x) q^x w^x [x]_q^k d\mu_{-1}(x).$$ \hspace{1cm} (22)

Therefore, by (22), we obtain the following theorem.

**Theorem 2.1.** Let $P(x) \in \mathbb{C}_p[[x]]_q$ be an arbitrary $q$-polynomial. Then $\tilde{\mu}_{P,-q}$ is a strongly weighted and twisted fermionic $p$-adic invariant measure on $\mathbb{Z}_p$. Then we have

$$f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x w^x P(x) \text{ for all } x \in \mathbb{Z}_p.$$  

Furthermore, for any $g \in UD(\mathbb{Z}_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) = \int_{\mathbb{Z}_p} g(x) P(x) q^x w^x d\mu_{-1}(x),$$

where the second integral is weighted and twisted fermionic $p$-adic invariant integral on $\mathbb{Z}_p$.

Let $f(x) = \sum_{n=0}^{\infty} a_{n,q} \binom{x}{n}_q$ be the Mahler $q$-expansion of continuous function on $\mathbb{Z}_p$, where

$$\binom{x}{n}_q = \frac{[x]_q [x-1]_q \cdots [x-n+1]_q}{[n]_q!}.$$

Then we note that $\lim_{n \to \infty} |a_{n,q}| = 0$.

Let

$$f_m(x) = \sum_{i=0}^{m} a_{i,q} \binom{x}{i}_q \in \mathbb{C}_p[[x]]_q.$$  

Then

$$\|(f - f_m)_q\|_{\infty} \leq \sup_{n \leq m} |a_{n,q}|.$$ \hspace{1cm} (23)
The function $f(x)$ can be rewritten as $f = f_m + f - f_m$. Thus, by (14) and (23), we get

$$\left| \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) - \hat{\mu}_{f,\infty}^w(a + p^n\mathbb{Z}_p) \right|_p \leq \max\{ \left| \hat{\mu}_{f,\infty}^w(a + p^n\mathbb{Z}_p) - \hat{\mu}_{f,\infty}^w(a + p^n\mathbb{Z}_p) \right|_p, \left| \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) - \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) \right|_p \}$$

From Theorem 2.1., we note that

$$\left| \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) \right|_p \leq \| f - f_m \|_\infty \leq C_1 p^{-2v_1(1-q^n)},$$

where $C_1$ are positive constants. For $m \gg 0$, we have $\| f \|_\infty = \| f_m \|_\infty$. So, we see that

$$\left| \hat{\mu}_{f,\infty}^w(a + p^n\mathbb{Z}_p) - \hat{\mu}_{f,\infty}^w(a + p^n\mathbb{Z}_p) \right|_p \leq \| f_m q^w x \|_\infty \| [p^n]_q^2 \| \leq C_2 p^{-2v_2(1-q^n)},$$

where $C_2$ is a positive constant.

By (25), we get

$$\left| (-1)^n q^w a^w f(a) - \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) \right|_p \leq \max\{ |q^n| a^w f(a) - f_m(a) q^n a^w \|_p, |q^n| a^w f_m(a) - \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) \|_p, \} \leq \max\{ |q^n| a^w f(a) - f_m(a) \|_p, |f_m(a) - \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) \|_p, \| (f - f_m) q^w x \|_\infty \}$$

Let us assume that fix $\epsilon > 0$, and fix $m$ such that $\| f - f_m \| < \epsilon$. Then we have

$$\left| (-1)^n q^w a^w f(a) - \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) \right|_p \leq \epsilon \quad \text{for} \quad n \gg 0.$$

Thus, by (28), we have

$$f_{\hat{\mu}_{f,-q}^w}(a) = \lim_{n \to \infty} \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) = (-1)^n q^w a^w f(a).$$

Let $m$ be the sufficiently large number such that $\| f - f_m \|_\infty \leq 15^{-n}$. Then we get

$$\hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) = \hat{\mu}_{f,\infty}^w(a + p^n\mathbb{Z}_p) + \hat{\mu}_{f,-q}^w(a + p^n\mathbb{Z}_p) = (-1)^n q^w a^w f(a) \pmod{[p^n]_q^2}.$$

For $g \in UD(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x) d\hat{\mu}_{f,-q}^w(x) = \int_{\mathbb{Z}_p} f(x) g(x) \frac{[2]_q}{1 + w q^w x} d\mu_{\mathbb{Z}_p}(x).$$

Let $f$ be the function from $UD(\mathbb{Z}_p)$ to $Lip(\mathbb{Z}_p)$. We easily see that $w^x q^w \mu_{\mathbb{Z}_p}(x + p^n\mathbb{Z}_p)$ is a strongly weighted and twisted $p$-adic invariant measure on $\mathbb{Z}_p$ and

$$\left| (f_{qw})_{\mu_{\mathbb{Z}_p}}(a) - w^a q^w a^w \mu_{\mathbb{Z}_p}(a + p^n\mathbb{Z}_p) \right|_p \leq C_3 p^{-2v_1(1-q^n)},$$

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where $f_{qw}(x) = q^x w^x f(x)$ and $C_3$ is positive constant and $n \in \mathbb{Z}_+$. 

If $\tilde{\mu}_{1-q}^w$ is associated with strongly weighted and twisted fermionic invarinat measure on $\mathbb{Z}_p$, then we have

$$\left| \tilde{\mu}_{1-q}^w(a + p^n\mathbb{Z}_p) - (f_{qw})_{\mu_{-1}}(a) \right|_p \leq C_4 p^{-2v_p(1-q^n)},$$

where $n > 0$ and $C_4$ is positive constant.

For $n \gg 0$, we have

$$\left| q^a w^a \mu_{-1}(a + p^n\mathbb{Z}_p) - \tilde{\mu}_{1-q}^w(a + p^n\mathbb{Z}_p) \right|_p \leq K,$$

where $K$ is positive constant.

Hence, $w^q \mu_{-1} - \tilde{\mu}_{1-q}^w$ is a weighted and twisted measure on $\mathbb{Z}_p$. Therefore, we obtain the following theorem.

**Theorem 2.2.** Let $w^q \mu_{-1}$ be a strongly weighted and twisted $p$-adic invariant measure on $\mathbb{Z}_p$, and assume that the fermionic weighted and twisted Radon–Nikodym derivative $(f_{qw})_{\mu_{-1}}$ on $\mathbb{Z}_p$ is uniformly differentiable function. Suppose that $\tilde{\mu}_{1-q}^w$ is the strongly weighted and twisted fermionic $p$-adic invariant measure associated with $(f_{qw})_{\mu_{-1}}$. Then there exists a weighted and twisted measure $\tilde{\mu}_{2-q}^w$ on $\mathbb{Z}_p$ such that

$$w^x q^x \mu_{-1}(x + p^n\mathbb{Z}_p) = \tilde{\mu}_{1-q}^w(x + p^n\mathbb{Z}_p) + \tilde{\mu}_{2-q}^w(x + p^n\mathbb{Z}_p).$$

### 3 Conclusion

The Theorem 2.2. is the version of the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted $p$-adic $q$-measure on $\mathbb{Z}_p$. In special case, if there is no twisted, then we can derive the same result as Jeong and Rim, 2012 (see [5]). In the case of weight zero and no twisted, then we derive the same result as Kim, 2012 (see [7]).

### References


