Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 20, 2014, No. 5, 49–57

On the local and global principle for system of binary rational cubic forms

Lan Nguyen

Department of Mathematics, University of Wisconsin-Parkside
Ann Arbor, MI 48109, United States
e-mail: nguyenl@uwp.edu

Abstract: It is known that any binary rational cubic form satisfies the Hasse principle. The next natural question to ask is whether this still holds for a system of binary rational cubic forms. However, there seems to be no known result on this topic. In our paper we show, by establishing an explicit equivalence between a rational cubic form and an intersection of quadric surfaces, that any system of finitely many binary rational cubic forms satisfies the Hasse principle.
Keywords: Hasse principle, Cubic plane curve, Cubic form, Quadratic form, System of cubic forms, System of binary quadratic forms, Selmer curve, Finite Basis theorem.
AMS Classification: 11XXX.

1 Introduction

1.1 Hasse Principle

One of the fundamental questions in Diophantine Number Theory is whether a rational Diophantine equation or a system of such equations has a solution in $\mathbb{Q}$. Answering such a question is difficult and not always possible. Such questions are addressed by Hilbert's 10th problem [5].

A solution to a Diophantine equation in t variables with coefficients in a field F is said to have a nontrivial solution $(\alpha_1, \ldots, \alpha_t)$ in F if α_j is in F for $1 \leq j \leq t$ and $\alpha_j \neq 0$ for at least one j. If a polynomial with rational coefficients has a nontrivial rational solution, then it has nontrivial solutions in all p-adic fields $\mathbb{Q}_p$ and in $\mathbb{R}$. The Hasse principle asks when the reverse direction is true. That is, if a polynomial with rational coefficients has a nontrivial solution in each p-adic field $\mathbb{Q}_p$ and in $\mathbb{R}$, does it have a nontrivial rational solution? If it does, then one says that it satisfies the Hasse principle. More generally, the Hasse principle for a variety is a statement about the existence of global points given the existence of local points.

For a quadratic form, Minkowski established around 1920 (generalized to arbitrary number fields by Hasse later) the following beautiful theorem [9]:

Theorem 1.1. *(Hasse-Minkowski Theorem) Let F be a quadratic form with rational coefficients. Then F has a nontrivial rational solution if and only if F has a nontrivial solution in each completion $\mathbb{Q}_p$ of $\mathbb{Q}$ where p ranges over all finite and infinite primes.*

For cubic forms, Selmer [8] shows, using the Finite Basis theorem of Mordell [6], that the Hasse principle fails with the following well-known counterexample:

$$3x^3 + 4y^3 + 5z^3 = 0.$$

H. Davenport [1] shows that the Hasse principle holds trivially for cubic forms with rational coefficients in at least 16 variables. Roger Heath-Brown proves a similar result, using the Hardy-Littlewood circle method, for cubic forms with rational coefficients in at least 14 variables [3]. For nonsingular cubic forms with rational coefficients, Hooley [4] uses a similar method to show that the Hasse principle holds for forms with 9 or more variables.

The main goal of this paper is to prove that any system of binary rational cubic forms in fact satisfies the Hasse principle. We obtain this result by establishing an explicit equivalence between a rational cubic plane curve (also a system of rational cubic plane curves) and an intersection of quadric surfaces.

2 Main results

Let us denote by $F_{u,v} = 0$ a system of v homogeneous forms, representing 0, of the same degree in u variables with rational coefficients. Then we say that $F_{u,v} = 0$ satisfies the Hasse principle if the forms in $F_{u,v} = 0$ has a common nontrivial rational solution if and only if they have a common nontrivial solution in every completion $\mathbb{Q}_p$ of $\mathbb{Q}$ and in $\mathbb{R}$.

Theorem 2.1. *Let*

$$f(x, y, z) = 0$$

be a cubic plane curve with rational coefficients. Then there exists a system $F_{6,9} = 0$ of 9 homogeneous quadratic forms in 6 variables, depending on $f(x, y, z) = 0$, with rational coefficients such that $f(x, y, z) = 0$ has nontrivial solutions in all completions $\mathbb{Q}_p$ of $\mathbb{Q}$ and in $\mathbb{R}$ if and only if

$$F_{6,9} = 0$$

has common nontrivial solutions in all completions $\mathbb{Q}_p$ of $\mathbb{Q}$ and in $\mathbb{R}$ and the Hasse principle for $f(x, y, z) = 0$ is equivalent to the Hasse principle for the system $F_{6,9} = 0$.

Theorem 2.2. *If*

$$\begin{cases} f_1(x,y,z) = 0 \\ f_2(x,y,z) = 0 \\ \vdots \\ f_m(x,y,z) = 0 \end{cases} \tag{2.1}$$

is a system of m homogeneous cubic forms with rational coefficients, then there exists a system $F_{6,3m+6} = 0$ of $3m+6$ homogeneous quadratic forms in 6 variables, depending on $f_i(x,y,z)$ for all $1 \le i \le m$, with rational coefficients such that system (2.1) has common nontrivial solutions in all completions $\mathbb{Q}_p$ of $\mathbb{Q}$ and in $\mathbb{R}$ if and only if $F_{6,3m+6} = 0$ has common nontrivial solutions in all completions $\mathbb{Q}_p$ of $\mathbb{Q}$ and in $\mathbb{R}$ and the Hasse principle for System (2.1) is equivalent to the Hasse principle of the system $F_{6,3m+6} = 0$.

Remark 2.3. Theorem 2.1 also provides a simple extra condition, from the perspective of quadratic forms, required for a cubic plane curve to satisfy the Hasse principle. That is, a cubic plane curve satisfies the Hasse principle if and only if a certain corresponding system of quadratic forms has the following equivalent properties:

- The existence of a common nontrivial solution in each completion $\mathbb{Q}_p$ and in $\mathbb{R}$ *implies* the existence of a common nontrivial rational solution.
- The existence of a nonempty intersection of the sets of nontrivial solutions in each completion $\mathbb{Q}_p$ and in $\mathbb{R}$ of each form in the system *implies* not only the set of nontrivial rational solutions of each form in the system is nonempty (Theorem 1.1) but also the existence of a nonempty intersection of these sets.

Theorem 2.1 allows us to obtain the following result concerning the Hasse principle of any system of finitely many binary cubic forms with rational coefficients.

Corollary 2.4. *Let*

$$\begin{cases} g_1(x,y) = 0 \\ g_2(x,y) = 0 \\ \vdots \\ g_n(x,y) = 0 \end{cases} \tag{2.2}$$

be a system of n homogeneous binary cubic forms with rational coefficients. Then System (2.2) satisfies the Hasse principle.

Remark 2.5.

1. It is known that any one homogeneous binary cubic form with coefficients in an algebraic number field satisfies the Hasse priciple [2].
2. Even though we are only interested in forms with rational coefficients, it should be noted that the proofs we provide below for Theorems 2.1 and 2.2 work for any field of characteristic zero. Our proof for Corollary 2.4 works for forms with fields of coefficients which are algebraic number fields.

3 Proof of results

Proof. (Proof of Theorem 2.1)
Let $F(x,y,z)=0$ be a general cubic plane curve with rational coefficients, i.e., $F(x,y,z)=0$ has the form

$$A_1x^3+A_2y^3+A_3z^3+A_4x^2y+A_5x^2z+A_6y^2x+A_7y^2z+ \tag{3.1}$$
$$+A_8z^2x+A_9z^2y+A_{10}xyz=0$$

where A_i is a rational number for each i. Suppose that $F(x,y,z)=0$ has a nontrivial solution in each $\mathbb{Q}_p$ and in $\mathbb{R}$.

In the rest of the paper, a local or global solution

$$(x_0,y_0,z_0)$$

of $F(x,y,z)=0$ is said to be nontrivial if at least one of the components in nonzero.

By multiplying both sides of (3.1) by x, y and z, we obtain respectively:

$$A_1x^4+A_2xy^3+A_3xz^3+A_4x^3y+A_5x^3z+A_6y^2x^2+A_7y^2zx+ \tag{3.2}$$
$$+A_8z^2x^2+A_9z^2yx+A_{10}x^2yz=0,$$
$$A_1x^3y+A_2y^4+A_3z^3y+A_4x^2y^2+A_5x^2zy+A_6y^3x+A_7y^3z+ \tag{3.3}$$
$$+A_8z^2xy+A_9z^2y^2+A_{10}xy^2z=0,$$
$$A_1x^3z+A_2y^3z+A_3z^4+A_4x^2yz+A_5x^2z^2+A_6y^2xz+A_7y^2z^2+ \tag{3.4}$$
$$+A_8z^3x+A_9z^3y+A_{10}xyz^2=0.$$

Let $X:=x^2$, $Y:=y^2$, $Z:=z^2$, $W:=xy$, $M:=xz$ and $N:=yz$. Then

$$WM-XN=0, \tag{3.5}$$
$$MN-ZW=0, \tag{3.6}$$
$$WN-YM=0, \tag{3.7}$$
$$W^2-XY=0, \tag{3.8}$$
$$M^2-XZ=0, \tag{3.9}$$
$$N^2-YZ=0. \tag{3.10}$$

Thus (3.2), (3.3) and (3.4) can be rewritten respectively as

$$A_1X^2+A_2YW+A_3ZM+A_4XW+A_5XM+A_6XY+A_7YM+ \tag{3.11}$$
$$+A_8XZ+A_9ZW+A_{10}XN=0,$$

$$A_1XW + A_2Y^2 + A_3ZN + A_4XY + A_5XN + A_6YW + A_7YN+ \tag{3.12}$$

$$+A_8ZW + A_9YZ + A_{10}YM = 0,$$

$$A_1XM + A_2YN + A_3Z^2 + A_4XN + A_5XZ + A_6YM + A_7YZ+ \tag{3.13}$$

$$+A_8ZM + A_9ZN + A_{10}ZW = 0.$$

Thus quadratic forms (3.5)-(3.13) have a common nontrivial solution in each $\mathbb{Q}_p$ and in $\mathbb{R}$ if $F(x, y, z) = 0$ is assumed to have a common nontrivial solution in each $\mathbb{Q}_p$ and in $\mathbb{R}$. For convenience, we denote (3.11), (3.12) and (3.13) by $F_x(X, Y, Z, W, M, N) = 0$, $F_y(X, Y, Z, W, M, N) = 0$ and $F_z(X, Y, Z, W, M, N) = 0$ respectively where each index indicates which variable is multiplied to both sides of (3.2) to form the corresponding equation.

Proposition 3.1. *The system of equations*

$$F_{6,9} = \begin{cases} F_x(X, Y, Z, W, M, N) = 0 \\ F_y(X, Y, Z, W, M, N) = 0 \\ F_z(X, Y, Z, W, M, N) = 0 \\ WM - XN = 0 \\ MN - ZW = 0 \\ WN - YM = 0 \\ W^2 - XY = 0 \\ M^2 - XZ = 0 \\ N^2 - YZ = 0 \end{cases} \tag{3.14}$$

has a common nontrivial solution in each $\mathbb{Q}_p$ and in $\mathbb{R}$ if and only if equation (3.1) has a nontrivial solution in each $\mathbb{Q}_p$ and in $\mathbb{R}$.

Proof. Let F denote either $\mathbb{R}$ or $\mathbb{Q}_p$ for some prime p. If equation (3.1) has a nontrivial solution in F, then it is clear from its construction that the system of equations (3.14) has a common nontrivial solution in F. Suppose that the system of equations (3.14) has a common nontrivial solution in F, say

$$(X_0, Y_0, Z_0, W_0, M_0, N_0) \neq (0, 0, 0, 0, 0, 0). \tag{3.15}$$

It follows from (3.11)-(3.13) that at least one of the three components X_0, Y_0 and Z_0 must be nonzero. If all three components are nonzero, then it can be verified that all the rest of the components must also be nonzero. If one of the components X_0, Y_0 and Z_0 is zero, then two of the components W_0, M_0 and N_0 must be zero. For example, if $X_0 = 0$, then it follows that $W_0 = 0$ and $M_0 = 0$. Now let us suppose that $X_0 \neq 0$. Then

$$\left(1, \frac{Y_0}{X_0}, \frac{Z_0}{X_0}, \frac{W_0}{X_0}, \frac{M_0}{X_0}, \frac{N_0}{X_0}\right) \tag{3.16}$$

is also a common nontrivial solution in F of system (3.16). Also,

$$\begin{cases} W_0^2 - X_0 Y_0 = 0 \\ M_0^2 - X_0 Z_0 = 0 \\ N_0^2 - Y_0 Z_0 = 0 \end{cases} \tag{3.17}$$

implies that

$$\begin{cases} \left(\dfrac{W_0}{X_0}\right)^2 = \dfrac{Y_0}{X_0} \\ \left(\dfrac{M_0}{X_0}\right)^2 = \dfrac{Z_0}{X_0} \\ \left(\dfrac{N_0}{X_0}\right)^2 = \dfrac{Y_0}{X_0}\dfrac{Z_0}{X_0}. \end{cases} \tag{3.18}$$

It is then straight forward, using (3.2), (3.7) and (3.13), to verify that

$$\left(1, \frac{W_0}{X_0}, \frac{M_0}{X_0}\right) \tag{3.19}$$

is a nontrivial solution in F of (3.1). Similar arguments work for $Y_0 \neq 0$ and for $Z_0 \neq 0$. Note that if $A_1 A_2 A_3 \neq 0$, then at most one in $\{X_0, Y_0, Z_0\}$ is 0. □

Proposition 3.2. *The system of equations*

$$F_{6,9} = \begin{cases} F_x(X,Y,Z,W,M,N) = 0 \\ F_y(X,Y,Z,W,M,N) = 0 \\ F_z(X,Y,Z,W,M,N) = 0 \\ WM - XN = 0 \\ MN - ZW = 0 \\ WN - YM = 0 \\ W^2 - XY = 0 \\ M^2 - XZ = 0 \\ N^2 - YZ = 0 \end{cases} \tag{3.20}$$

has a common nontrivial rational solution if and only if equation (3.1) has a nontrivial rational solution.

Proof. The same proof as above works, with F now denoting $\mathbb{Q}$. □

Theorem 2.1 follows immediately from Propositions 3.1 and 3.2. □

Proof. (Proof of Theorem 2.2)

Proposition 3.3. *Let*

$$\begin{cases} f_1(x,y,z) = 0 \\ f_2(x,y,z) = 0 \\ \vdots \\ f_m(x,y,z) = 0 \end{cases} \tag{3.21}$$

be a system of m cubic plane curves with rational coefficients. Define

$$F(6, 3m+6) := \begin{cases} F_x^{(1)}(X,Y,Z,M,N,W) = 0 \\ F_y^{(1)}(X,Y,Z,M,N,W) = 0 \\ F_x^{(1)}(X,Y,Z,M,N,W) = 0 \\ \vdots \\ F_x^{(m)}(X,Y,Z,M,N,W) = 0 \\ F_y^{(m)}(X,Y,Z,M,N,W) = 0 \\ F_x^{(m)}(X,Y,Z,M,N,W) = 0 \\ WM - XN = 0 \\ MN - ZW = 0 \\ WN - YM = 0 \\ W^2 - XY = 0 \\ M^2 - XZ = 0 \\ N^2 - YZ = 0 \end{cases} \tag{3.22}$$

where X, Y, Z, M, N and W are defined as above while

$$F_x^{(i)}(X,Y,Z,M,N,W) := xf_i(x,y,z),$$

$$F_y^{(i)}(X,Y,Z,M,N,W) := yf_i(x,y,z),$$

and

$$F_z^{(i)}(X,Y,Z,M,N,W) := zf_i(x,y,z)$$

for $1 \leq i \leq m$. Then:

1. *System (3.21) has a common nontrivial solution in each $\mathbb{Q}_p$ and in $\mathbb{R}$ if and only if the system (3.22) does.*

2. *System (3.21) has a common nontrivial solution in $\mathbb{Q}$ if and only if the system (3.22) does.*

Proof. The proof is a generalization of that of Theorem 2.1.

(1) Let F denote either $\mathbb{R}$ or $\mathbb{Q}_p$ for some prime p. From construction, it is also clear that if system (3.21) has a common nontrivial solution in F, then system (3.22) does as well. For the other direction, the proof of Theorem 2.1 can be modified as follows: The nontrivial solutions

(3.15) and (3.16) in the proof of Theorem 2.1 are now nontrivial solutions of system (3.22) in F. Systems (3.17) and (3.18) are exactly the same for this case. The solution (3.19) in the proof of Theorem 2.1 can now be shown to be a common nontrivial solution of system (3.21) in F.

(2) The same argument as in (1) works, with F denoting $\mathbb{Q}$. □

By Prop 3.3, Theorem 2.2 follows. □

Proof. (Proof of Corollary 2.4)

Let

$$\begin{cases} g_1(x,y) = 0 \\ g_2(x,y) = 0 \\ \vdots \\ g_n(x,y) = 0 \end{cases} \tag{3.23}$$

be a system of n homogeneous binary cubic forms with rational coefficients. Let X, Y and W be defined as before. Define

$$\begin{cases} F_x^{(1)}(X,Y,W) = 0 \\ F_y^{(1)}(X,Y,W) = 0 \\ \vdots \\ F_x^{(n)}(X,Y,W) = 0 \\ F_y^{(n)}(X,Y,W) = 0 \\ W^2 - XY = 0 \end{cases} \tag{3.24}$$

where

$$F_x^{(i)}(X,Y,Z) := xg_i(x,y)$$

and for $1 \le i \le n$

$$F_y^{(i)}(X,Y,Z) := yg_i(x,y).$$

Proposition 3.4. *1. System (3.23) has a common nontrivial solution in $\mathbb{R}$ or $\mathbb{Q}_p$ for any prime p if and only if system (3.24) does.*

2. System (3.23) has a common nontrivial solution in $\mathbb{Q}$ if and only if system (3.24) does.

Proof. (1) For each i, the form $g_i(x,y) = 0$ in (3.23) takes the form

$$A_1x^3 + A_2y^3 + A_3z^3 + A_4x^2y + A_5x^2z + A_6y^2x + A_7y^2z + A_8z^2x + A_9z^2y + A_{10}xyz = 0$$

with each A_j being rational and with

$$A_3 = A_5 = A_7 = A_8 = A_9 = A_{10} = 0.$$

Therefore, the proof of (1) of Theorem 2.2 applies and the result follows.

(2) This follows from the same argument as in (1) but uses the proof of (2) of Theorem 2.2 instead. □

Proposition 3.5. *System (3.24) satisfies the Hasse principle.*

Proof. In Theorem 1 of [7], A. Schinzel proves that any system of finitely many ternary rational quadratic forms satisfies the Hasse principle. Therefore, System (3.24) satisfies the Hasse Principle as required. □

Corollary 2.4 follows then from Propositions 3.4 and 3.5. □

References

[1] Davenport, H., Cubic Forms in 16 Variables, *Proceedings of the Royal Society A*, Vol. 272, 1963, 285–303.

[2] Fujiwara, M., Hasse Principle in Algebraic Equations, *Acta Arith.*, Vol. 22, 1973, 267–276.

[3] Heath-Brown, D. R., Cubic Forms in 14 Variables, *Invent. Math.*, Vol. 170, 2007, No. 1, 199–230.

[4] Hooley, C., On Nonary Cubic Forms, *J. Für Die Reine Und Angewandte Mathematik*, Vol. 386, 1988, 32–98.

[5] Matiyasevich, Y., *Hilbert's Tenth Problem*, MIT Press, Massachusetts, 1993.

[6] Mordell, L., On the Rational Solutions of the Indeterminate Equations of the Third and Fourth Degrees, *Proc. Cambridge Phil. Soc.*, Vol. 21, 1922–1923, 179–192.

[7] Schinzel, A., Hasse's Principle for Systems of Ternary Quadratic Forms and for one Biquadratic form, *Studia Mathematica*, Vol. LXXVII, 1983, 103–109.

[8] Selmer, E., The Diophantine Equation $ax^3 + by^3 + cz^3$, *Acta Math.*, Vol. 85, 1951, No. 1, 203–362.

[9] Serre, J. P., *A First Course in Arithmetic*, Graduate Texts in Mathematics, Vol. 7, Berlin, New York: Springer–Verlag, 1973.