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Some results on self vertex switching

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Abstract: Let $G(V, E)$ be a graph. A vertex $v \in V(G)$ is said to be a self vertex switching of G, if G is isomorphic to G^v, where G^v is the graph obtained from G, by deleting all edges of G incident to v and adding edges between v and the vertices which are not adjacent to v in G. In this paper, we discuss some applications of self vertex switching and list out all trees and unicyclic graphs with unique self vertex switching. We also obtain some more results on self vertex switching.

Keywords: Switching, Self vertex switching, Trees, Unicyclic graphs.
AMS Classification: 05CXX.

1 Introduction

Throughout this paper, we consider only finite, simple, undirected graphs. For notations and terminology, we follow [1]. The *eccentricity* of a vertex v, $e(v)$, is defined to be the distance of a farthest vertex from v. A vertex v is said to be an *eccentric vertex* of u, if $d(u, v) = e(u)$. The *diameter*, diam(G), of a graph G is the maximum eccentricity and the *radius*, $r(G)$, of G is the minimum eccentricity in G. A vertex v is said to be a *central vertex* of G if $e(v) = r(G)$. For any vertex $v \in V(G)$, the *open neighbourhood* of v is the set of all vertices adjacent to v. That is, $N(v) = \{u \in V(G) \,/\, uv \in E(G)\}$. The *closed neighbourhood* of v is defined by $N[v] = N(v) \cup \{v\}$. The set $V \setminus N[v]$ is denoted by $N[v]^c$ and the join of two graphs G_1 and G_2 is denoted by $G_1 \vee G_2$. Let nG denote the disjoint union of n copies of G.

Normally, while analysing the properties of a graph, the neighbour set of each vertex is considered to be a constant set or a set which adds additional elements. That is, a neighbour of a vertex remains its neighbour forever. But in real life, this is not proved to be true always. Even no relationship in this world proves itself to be eternal now-a-days. So it is no longer of greater use to consider the neighbour set of a node in a system to be rigid and constant. What happens if a vertex in a graph changes its neighbour often?

Consider the following real life problems:

1. There are many official networks in which we can classify the nodes into units as per their work nature. The work distribution will not be even among all the units. The

efficiency of any unit of heavy duty will be greatly reduced, if it continues to work without break. It would not be economically advisable to design a duplicate network of the whole system to provide a substitution for this unit during its recharging period. Within the same system, it would be better if there is work exchange without affecting the overall structure of the network.

2. In a confidential system of data transaction, the data supplier should not be identified easily. For that, he should not be in fixed rigid contact with his primary data collectors. But the whole system should not have any change in its outlook to avoid any suspicion. Within the same network, it would be better if there is a dynamical vertex, switching over of which should not affect the structure of the system.
3. While dealing with power cut, it would be better if there is a two way process in the same unit, which provides extra facilities along with the primary work when there is power supply and mere necessary output when there is power cut. But there cannot be separate original and duplicate network as it is not economical. A single unit should provide a two way process.

All the above problems can be solved, if we can find a node (vertex) in the network models (graphs) which changes its neighbours as required without affecting the structure of the whole system. The problem of designing systems with such a node is equivalent to the problem of constructing a graph with a self vertex switching, the definition of which is given as follows:

The switching concept was introduced by Seidel in [5]. For a graph $G(V, E)$ and a subset S of V, the *switching of G by S* is defined as the graph $G^S(V, E')$, which is obtained from G, by removing all edges between S and its complement $V \setminus S$ and adding edges between S and $V \setminus S$ which are not in G. For example, a graph G with $S = \{v_1, v_2\}$ and G^S are shown in Fig. 1.

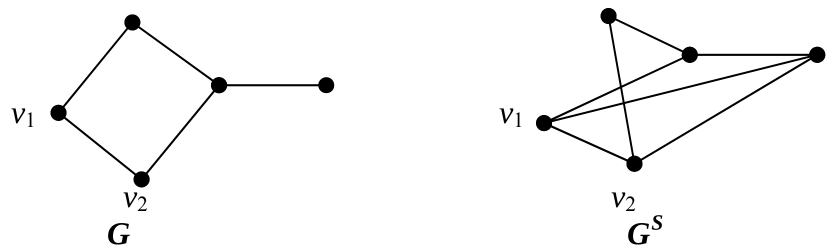


Figure 1

When $S = \{v\}$, the corresponding switching is called *vertex switching* and it is denoted by G^v. In this case, we call the vertex v as a vertex switching. For example, a graph G and a vertex switching v of G are shown in Fig. 2.

Figure 2

A subset S of $V(G)$ is said to be a *self switching* of G if $G \cong G^S$. The set of all self switchings of G with cardinality k is denoted by $SS_k(G)$ and the number of self switchings of G

with cardinality k is denoted by $ss_k(G)$. If $k = 1$, then the particular self switching is called a self vertex switching. A graph G with a self vertex switching v is shown in Fig. 3.

For further details on self vertex switchings, one can refer [2], [3], [4] and [6]. It has been proved in [4] that a graph of order $2n + 1$ has at most $n + 1$ self vertex switchings. Characterisation of graphs of order $2n + 1$ with $n + 1$ mutually non-adjacent self vertex switchings and that with mutually adjacent self vertex switchings are given in [4].

$\boldsymbol{G}$ $\qquad$ $\boldsymbol{G^v}$

Figure 3

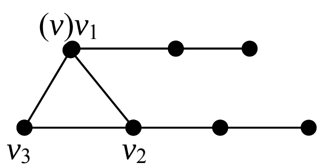
In fact, it is proved that $K_{n,n+1}$ is the only graph of order $2n + 1$ in which there are $n + 1$ mutually non adjacent self vertex switchings whereas $K_n \cup K_{n+1}$ is the only graph of order $2n + 1$ in which there are $n + 1$ mutually adjacent self vertex switchings. Characterisation of a cut vertex in a connected graph to be a self vertex switching has been given in [6].

The *girth* of a graph G is defined to be the length of a smallest cycle in G and the *circumference* of G is the length of a longest cycle in G. In a graph G, deleting an edge uv and introducing a new vertex w and the new edges uw and vw is called the *subdivision of the edge uv*. The *edge subdivision graph* denoted by $S_1(G)$ is obtained from the graph G by subdividing every edge of G. For example, $S_1(K_{1,5})$ is shown in Fig. 4.

$\boldsymbol{K_{1,5}}$ $\qquad$ $\boldsymbol{S_1(K_{1,5})}$

Figure 4

Let $C_r(v)$ denote the cycle $v_1v_2....v_rv_1$ of order r with a fixed vertex $v = v_1$ and let $C_r(v)(P_{n_1}, P_{n_2}, \ldots, P_{n_r})$ be the graph obtained from $C_r(v)$ by identifying an end vertex of the path P_{n_i} with the vertex v_i. For example, $C_3(v)(P_3, P_3, P_1)$ is shown in Fig. 5.



$C_3(v)(P_3, P_3, P_1)$

Figure 5

Two vertices u and v in G are said to be *interchange similar* [3], if there exists an automorphism α of G such that $\alpha(u) = v$ and $\alpha(v) = u$.

For further investigation, we need the following results.

Theorem A [6]: If v is a self vertex switching of a graph G of order p, then $d_G(v) = (p - 1)/2$.

Theorem B [6]: Let G be a graph in which any two self vertex switchings, if exist, are interchange similar. A vertex v in G is a self vertex switching if and only if $G - v$ has an automorphism, which maps the elements of $N(v)$ onto $N(v)^c$.

That is, for any graph G with v as a self vertex switching, $\langle N(v)\rangle \cong \langle N[v]^c\rangle$.

Theorem C [6]: Let G be a connected graph with a cut vertex as a self vertex switching such that any two self vertex switchings, if exist, are interchange similar. Then $ss_1(G) > 1$ if and only if $G \cong C_3(v)(P_3, P_1, P_3)$.

In this paper, we characterise trees and unicyclic graphs that have a unique self vertex switching. We construct graphs having a self vertex switching with given radius or circumference or diameter. Also we prove that any graph G is an induced subgraph of a graph H with a self vertex switching $\{v\}$ such that v is a central vertex of H with eccentricity $r(G) + 1$.

2 Main results

The following theorem characterises all connected graphs of girth greater than four, which have a unique self vertex switching.

Theorem 1. A connected graph G with girth at least five has $ss_1 = 1$ if and only if $G \cong S_1(K_{1,n})$.

Proof: Let v be the self vertex switching of G. Then we have $G \cong G^v$ and so G^v also has girth greater than or equal to five. Now, let $V(G) = N(v) \cup (N(v))^c$. By Theorem B, $\langle N(v)\rangle \cong \langle N[v]^c\rangle$. But $N[v]^c$ must be an independent set in G, otherwise G^v contains a triangle, which is impossible. This means that $\langle N(v)\rangle$ is a null graph in G. Therefore for any edge xy in G, we have $x \in N(v)$ and $y \in N(v)^c$. Since G^v is connected, no vertex in $N(v)$ is of degree 1. This forces that every vertex in $N(v)$ is of degree exactly two, otherwise G^v contains a C_4. Similarly if a vertex in $N[v]^c$ is of degree greater than one, then G contains a C_4, which is impossible. Thus $d(u) = 1$, for all $u \in N[v]^c$. Such a graph G is isomorphic to $S_1(K_{1,n})$.

The converse is obvious. □

Since the girth of any tree is infinite, we can state that

Corollary 1.1. For any tree T, $ss_1 = 1$ if and only if $T \cong S_1(K_{1,n})$.

For any unicyclic graph G, girth is the length of the unique cycle. And if the girth is greater than four and $ss_1 = 1$, then by the above theorem, $G \cong S_1(K_{1,n})$, which is acyclic. This is a contradiction. Hence, we can state that:

Corollary 1.2. If a connected unicyclic graph G has $ss_1 = 1$, then the unique cycle in G is of length 3 or 4.

Let $G_1(n)$ denote the graph obtained by identifying the central vertex of $S_1(K_{1,n})$ with v in $C_3(v)(P_4, P_2, P_1)$ and $G_2(n)$ denote the graph obtained by identifying the central vertex of $S_1(K_{1,n})$ and a pendant vertex of $K_{1,3}$ with v in $C_4(v)$. For example, $G_1(3)$ and $G_2(2)$ are shown in Fig. 6.

Figure 6

Now we give a characterisation for a unicyclic graph with unique self vertex switching.

Theorem 2. For a connected unicyclic graph G, $ss_1 = 1$ if and only if $G \cong G_1(n)$ or $G_2(n)$.

Proof: Assume that $ss_1 = 1$ and let v be the self vertex switching in G. Then we have $G \cong G^v$ and $\langle N(v)\rangle \cong \langle N[v]^c\rangle$. Let C be the unique cycle in G.

Case (i): $\langle N(v)\rangle$ contains an edge uw.

Then $uvwu$ is a triangle in G. Since G is unicyclic, uw is the unique edge in $N(v)$. Also since $\langle N(v)\rangle \cong \langle N[v]^c\rangle$, there exist two adjacent vertices u_1 and w_1 in $N[v]^c$. Since G is connected and unicyclic, exactly one of these two vertices u_1 and w_1 is adjacent to a vertex, say x, in $N(v)$. By isomorphism, exactly one of the vertices u and w, is adjacent to a vertex, say y, in $N[v]^c$. Since $G - \{u, w, u_1, w_1, x, y\}$ is a tree with a self vertex switching, by Corollary 1.1, $\langle G - \{u, w, u_1, w_1, x, y\}\rangle \cong S_1(K_{1,n})$. Then it is clear that G is isomorphic to $G_1(n)$.

Case (ii): $\langle N(v)\rangle$ contains no edge.

Then $\langle N[v]^c\rangle$ is also a null graph. If $V(C)$ is contained in $N(v) \cup N[v]^c$, then G contains at least two cycles containing v, which is a contradiction.

Therefore, $v \in V(C)$. Since $N(v)$ is an independent set and by Corollary 1.2, the cycle is of length 4. Hence the cycle contains the vertex v, two neighbours of v and a non-neighbour of v. Since a vertex in $N[v]^c$ is adjacent to two vertices in $N(v)$, there will be a vertex in $N(v)$, which is adjacent to two vertices of $N[v]^c$. The resultant graph is isomorphic to $G_2(n)$.

And the converse is obvious, since v is clearly the self vertex switching in $G_1(n)$ or $G_2(n)$. □

Theorem 3. For any $n \geq 4$, there exists a graph with a self vertex switching and circumference equal to n.

Proof: Suppose n is even. Then $n = 2m$ for some positive integer $m \geq 2$. In this case, consider the cycle $C_{2m} = u_1\ u_2\ \ldots\ u_{2m}$. Now construct a graph G_{2m} with $V(G_{2m}) = \{v\} \cup V(C_{2m})$ and $E(G_{2m}) = E(C_{2m}) \cup \{vu_{2i} \mid 1 \leq i \leq m\}$. Then clearly circumference of G is $2m$. And v is obviously the self vertex switching in G_{2m} and hence G_{2m} is the required graph.

On the other hand, if n is odd, then $n = 2m + 1$, $m \geq 2$. Now construct a graph G_{2m+1} with $V(G_{2m+1}) = \{v\} \cup V(C_{2m})$ and $E(G_{2m+1}) = E(C_{2m}) \cup \{vu_i \mid 1 \leq i \leq m\}$. Clearly the circumference of G_{2m+1} is $2m + 1$ and so G is the required graph with a self vertex switching v.

For example, the graphs G_6 and G_7 are shown in Fig. 7. Here v is the self vertex switching of G_6 and v_1, v_2 are the self vertex switchings of G_7.

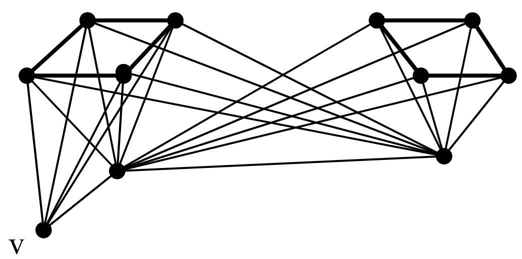
Figure 7

Note that for n = 5, 6 and $n \geq 8$, the graph G_n constructed above has unique self vertex switching whereas the graphs G_4 and G_7 have two self vertex switchings. □

Theorem 4. Every graph is an induced subgraph of a graph H with $ss_1 = 1$ and $r(H) = 1$.

Proof: Let G be any graph. Consider the graph $K_2 \vee 2G$. Add a new vertex v and join v with all vertices in one copy of G and a vertex of K_2. The graph thus formed is the required graph H with v as the unique self vertex switching. Since H has a full vertex, $r(H) = 1$. Note that the graph H is nothing but $2(G \vee K_1) \vee K_1$.

For example, the graph H having $G = C_4$ as an induced subgraph with $r(H) = ss_1(H) = 1$ is shown in Fig. 8. Here v is the unique self vertex switching. □



G

Figure 8

Theorem 5. Any connected graph G is an induced subgraph of a graph H with a self vertex switching v such that v is a central vertex of H with eccentricity $r(G) + 1$.

Proof: Let G be any connected graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$. Let u_m be any central vertex of G and let u_r be an eccentric vertex of u_m. Take four copies G_1, G_2, G_3, G_4 of G with $V(G_1) = \{v_1, v_2, \ldots, v_n\}$, $V(G_2) = \{w_1, w_2, \ldots, w_n\}$, $V(G_3) = \{x_1, x_2, \ldots, x_n\}$ and $V(G_4) = \{y_1, y_2, \ldots, y_n\}$ such that u_i corresponds to the vertices v_i, w_i, x_i, y_i in the respective copy. Now construct a graph H with $V(H) = V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4) \cup \{v\}$ and $E(H) = E(G_1) \cup E(G_2) \cup E(G_3) \cup E(G_4) \cup \{vv_j, vw_j \mid 1 \leq j \leq n, j \neq m\} \cup \{vx_m, vy_m\}$.

Then clearly $\langle N(v)\rangle \cong \langle N[v]^c\rangle$. Therefore $\{v\}$ is a self vertex switching of H.

Also by construction, v is in every path connecting a vertex and its eccentric vertex. Therefore, v is a central vertex of H. All v_j and w_j $(1 \le j \le n)$ are at a distance at most 2 from v. Let x_ℓ be an eccentric vertex of v. Then $d(v, x_\ell) = 1 + d(x_m, x_\ell)$. This distance is maximum when $x_\ell = x_r$ and is equal to $r(G)$. Therefore $r(H) = r(G)+1$.

Hence H is the required graph containing G as an induced sub graph. For example, a graph G and the corresponding graph H constructed above are shown in Fig. 9. ☐

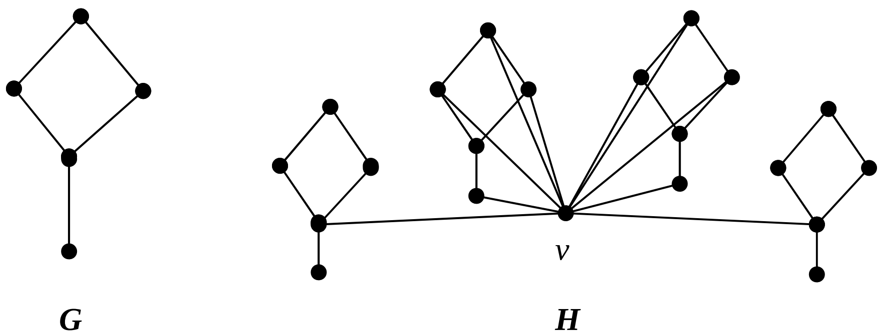


Figure 9

Theorem 6. For any $n \ge 2$, there exists a graph G of diameter n with a self vertex switching.

Proof: For $n = 2$, 3 or 4, the required graph G_n with diameter n and a self vertex switching v is shown in Fig. 10. Therefore, assume that $n \ge 5$. Consider the path $P_{2(n-2)} = u_1\ u_2\ \dots\ u_{2(n-2)}$. Now construct the graph G_n with $V(G) = \{v\} \cup V(P_{2(n-2)})$ and $E(G_n) = E(P_{2(n-2)}) \cup \{vu_i \mid 1 \le i \le n-2\}$. Then clearly diameter of G is n. And v is the only self vertex switching of G_n and hence G_n is the required graph.

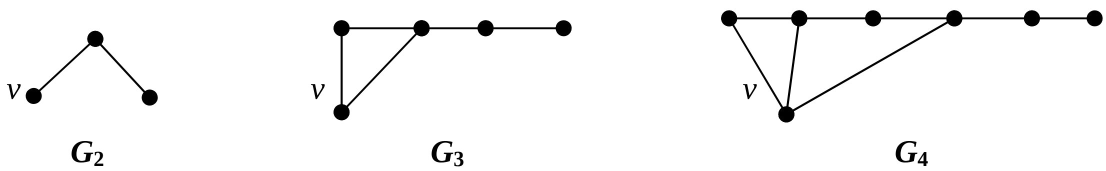


Figure 10

The graphs G_5 and G_6 constructed in the above theorem are given in Fig. 11. Here v_1 and v_2 are the self vertex switchings of G_5 whereas v is the self vertex switching of G_6. ☐

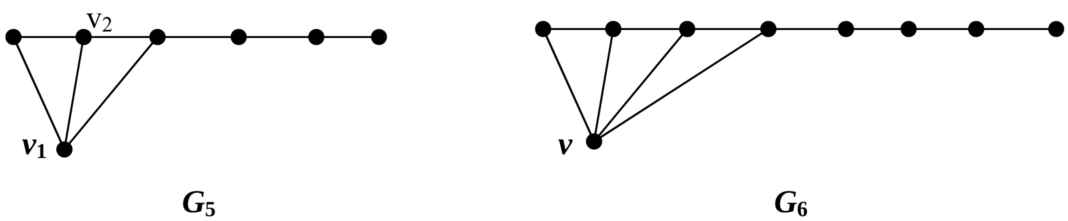


Figure 11

Note that for $n \ge 6$, the graph constructed in the above theorem will have unique self vertex switching.

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