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Arithmetic progressions of rectangles on a conic

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Abstract: In this paper we find infinitely many parabolas on which there exist five points with integer co-ordinates $(x_j, y_j),\ j = 1, 2, \ldots, 5$, such that the products $x_j y_j,\ j = 1, 2, \ldots, 5$, are in arithmetic progression. Similarly, we find infinitely many ellipses and hyperbolas on which there exist six points with integer co-ordinates $(x_j, y_j),\ j = 1, 2, \ldots, 6$, such that the products $x_j y_j,\ j = 1, 2, \ldots, 6$, are in arithmetic progression. Brown had conjectured that there cannot exist four points with integer co-ordinates $(x_j, y_j),\ j = 1, 2, 3, 4$, on a conic such that the four products $x_j y_j,\ j = 1, 2, 3, 4$, are in arithmetic progression. The results of this paper disprove Brown's conjecture.
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1 Introduction

It was known to Fermat that there cannot exist four squares in arithmetic progression. A proof of this fact was first published by Euler and subsequently the proposition was proved by other mathematicians as well (see Dickson [2, p. 440]). Brown [1] has pointed out that the non-existence of four squares in arithmetic progression can be expressed by saying that there cannot exist four points with integer co-ordinates $(x_j, y_j),\ j = 1, 2, 3, 4$, on the line $y = x$ such that the four products $x_j y_j,\ j = 1, 2, 3, 4$, are in arithmetic progression. Starting with this observation, he has generalised the problem to finding four points with integer co-ordinates $(x_j, y_j),\ j = 1, 2, 3, 4$, on an arbitrary straight line, or on a conic, such that the four products $x_j y_j,\ j = 1, 2, 3, 4$, (that is, the areas of the four rectangles whose vertices are $(0, 0),\ (x_j, 0),\ (0, y_j),\ (x_j, y_j)$, $j = 1, 2, 3, 4$) are in arithmetic progression. He has established that there do not exist four such points on the line $y = ax + b$ where a, b are rational and $a \neq 0$. He has also proved that four such points cannot exist on a circle. Brown has conjectured that four such points cannot lie on a parabola or on the general conic defined by the second degree equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{1}$$

where the coefficients a, b, $\ldots$, f, are rational numbers.

This paper is concerned with the generalised problem posed by Brown. It is shown in Section 2 that there exist infinitely many parabolas on which there exist five points with integer co-ordinates $(x_j,\, y_j),\; j = 1, 2, 3, 4, 5$, such that the five products $x_jy_j,\; j = 1, \ldots, 5$, are in arithmetc progression. Further, in Section 3, it is shown that there exist infinitely many ellipses and hyperbolas on which there exist six points with integer co-ordinates $(x_j,\, y_j),\; j = 1, 2 \ldots, 6$, such that the six products $x_jy_j,\; j = 1, 2 \ldots, 6$, are in arithmetic progression.

We note that if there exist n points with rational co-ordinates $(x_j,\, y_j),\; j = 1, \ldots, n$, on the conic (1) such that the products $x_jy_j,\; j = 1, \ldots, n$, are n rational numbers in arithmetic progression, we can choose suitable integers $p,\; q$ and make a linear transformation $x = X/p$, $y = Y/q$, so that in the X–Y plane, the points $(X_j,\, Y_j) = (px_j,\, qy_j),\; j = 1, \ldots, n$, on the resulting conic have integer co-ordinates and are such that the n products $X_jY_j = pqx_jy_j$, $j = 1, \ldots, n$, are in arithmetic progression. It therefore suffices to find rational points $(x_j,\, y_j)$ on the conic (1) such that the products x_jy_j are in arithmetic progression.

2 Arithmetic progressions of five rectangles on a parabola

We initially assume that the parabola is defined, in terms of a parameter m, by the following equations:

$$x = a_0m^2 + a_1m + a_2, \quad y = b_0m^2 + b_1m + b_2, \tag{2}$$

where $a_0,\, a_1,\, a_2,\, b_0,\, b_1, b_2$ are rational coefficients to be determined. To find five points $(x_j,\, y_j)$ on the parabola such that the products x_jy_j are in arithmetic progression, we may take the five points defined by $m = m_j,\; j = 1, \ldots, 5$, where m_j are arbitrary rational parameters, and impose the requirement that the products x_jy_j are given by $p - 2q$, $p - q$, p, $p + q$ and $p + 2q$ where $p,\; q$ are arbitrary rational parameters. This gives us the following conditions:

$$\begin{aligned}
(a_0m_1^2 + a_1m_1 + a_2)(b_0m_1^2 + b_1m_1 + b_2) &= p - 2q,\\
(a_0m_2^2 + a_1m_2 + a_2)(b_0m_2^2 + b_1m_2 + b_2) &= p - q,\\
(a_0m_3^2 + a_1m_3 + a_2)(b_0m_3^2 + b_1m_3 + b_2) &= p,\\
(a_0m_4^2 + a_1m_4 + a_2)(b_0m_4^2 + b_1m_4 + b_2) &= p + q,\\
(a_0m_5^2 + a_1m_5 + a_2)(b_0m_5^2 + b_1m_5 + b_2) &= p + 2q.
\end{aligned} \tag{3}$$

Equations (3) may be considered as five homogeneous linear equations in the variables b_0, b_1, b_2, p, q, and the condition of solvability of these linear equations is easily worked out. This condition of solvability turns out to be a linear equation in the variable a_2 and is hence readily solved for a_2. With this value of a_2, equations (3) can be solved to obtain a parabola (2) whose coefficients $a_0,\, a_1,\, a_2,\, b_0,\, b_1, b_2$ are in terms of the parameters $a_0,\, a_1$ as well as the parameters $m_j,\, j = 1, 2, 3, 4, 5$. The values of the coefficients so obtained are too cumbersome to be written explicitly. It suffices for our purpose to assign the values -2, -1, 0, 1, to the parameters m_1, m_2, m_3, m_4 respectively and thus obtain a simpler solution. With these values of m_j, we readily obtain the following solution of the first four equations of (3):

$$\begin{array}{rcl} b_0 &=& a_0^2, \;\; b_1 \;=\; (2a_0 - a_1)a_0, \;\; b_2 \;=\; -a_0^2 - 2a_0a_1 + a_1^2 - a_0a_2, \\ p &=& -a_2(a_0^2 + 2a_0a_1 - a_1^2 + a_0a_2), \\ q &=& (a_0 - a_1)(2a_0^2 + a_0a_1 - a_1^2 + 2a_0a_2). \end{array} \tag{4}$$

On substituting these values in the last equation of (3), we get an equation in which a_2 occurs only in the first degree and we thus obtain the following solution for a_2:

$$\begin{array}{c} a_2 = -\{(m_5^4 + 2m_5^3 - m_5^2 - 4)a_0^3 - (m_5 - 2)a_0^2a_1 - (2m_5 - 4)a_0a_1^2 \\ +(m_5 - 2)a_1^3\}/\{2a_0(a_0 - a_1)(m_5 - 2)\}. \end{array} \tag{5}$$

It follows that when the coefficients of the parabola (2) are defined by (4) and (5) in terms of arbitrary rational parameters a_0, a_1 and m_5, the five points on the parabola, obtained by taking the parameter m as $-2, -1, 0, 1$ and m_5, have rational co-ordinates such that the products x_jy_j, $j = 1, \ldots, 5$, are in arithmetic progression.

As a numerical example, taking $a_0 = -1$, $a_1 = 1$, $m_5 = 3$, and using (4) and (5), we get the values of the coefficients as $a_2 = 30$, $b_0 = 1$, $b_1 = 3$, $b_2 = 32$, so that our parabola is given by the equations

$$x = -m^2 + m + 30, \quad y = m^2 + 3m + 32, \tag{6}$$

while the five points are obtained by assigning to m successively the values $-2, -1, 0, 1$ and 3. We thus get the five points (x_j, y_j) as $(24, 30), (28, 30), (30, 32), (30, 36), (24, 50)$ which are such that the products x_jy_j, namely $720, 840, 960, 1080, 1200$, are in arithmetic progression. Eliminating m between the two equations (6), we get the equation of our parabola in the standard form (1). We thus get the parabola

$$x^2 + 2xy + y^2 - 112x - 128y + 3612 = 0 \tag{7}$$

and it is readily verified that the aforementioned five points lie on this parabola.

We can get infinitely many numerical examples of parabolas with five points such that the areas of the corresponding rectangles are in arithmetic progression by starting with different numerical values of the parameters a_0, a_1, m_j, $j = 1, 2, 3, 4, 5$. In fact, it is conceivable that there exist parabolas on which we can get six, or perhaps even seven, points (x_j, y_j) such that the products x_jy_j are in arithmetic progression. The equations, however, get cumbersome and we could not find any such example.

3 Arithmetic progressions of six rectangles on an ellipse or a hyperbola

In order to find arithmetic progressions of six rectangles on a conic, we assume that there exist six points (x_j, y_j) on the conic (1) given by $((p-5q)r, st)$, $((p-3q)s, rt)$, $((p-q)t, rs)$, $((p+q)t, rs)$, $((p+3q)s, rt)$ and $(p+5q)rs, t)$ so that the products x_jy_j are $(p-5q)rst$, $(p-3q)rst$, $(p-q)rst$, $(p+q)rst$, $(p+3q)rst$, $(p+5q)rst$ and are hence in arithmetic progression. Substituting the co-ordinates of these six points in equation (1) in turn, we get the following six conditions that must be satisfied for these points to lie on the conic:

$$
\begin{array}{rcl}
(p-5q)^2r^2a+(p-5q)rstb+s^2t^2c+(p-5q)rd+ste+f &=& 0,\\
(p-3q)^2s^2a+(p-3q)rstb+r^2t^2c+(p-3q)sd+rte+f &=& 0,\\
(p-q)^2t^2a+(p-q)rstb+r^2s^2c+(p-q)td+rse+f &=& 0,\\
(p+q)^2t^2a+(p+q)rstb+r^2s^2c+(p+q)td+rse+f &=& 0,\\
(p+3q)^2s^2a+(p+3q)rstb+r^2t^2c+(p+3q)sd+rte+f &=& 0,\\
(p+5q)^2r^2s^2a+(p+5q)rstb+t^2c+(p+5q)rsd+te+f &=& 0.
\end{array} \tag{8}
$$

These are six homogeneous linear equations in the variables a, b, c, d, e, f and on solving the first five of these equations, we get the following solution:

$$
\begin{array}{rcl}
a &=& (r-s)(s-t)(t-r)rst,\\
b &=& 2(r-s)(s-t)(t-r)pst,\\
c &=& (p-5q)^2r^3s-(p-5q)^2r^3t-2p(p-5q)r^2s^2\\
&& \quad +2p(p-5q)r^2t^2+(p^2-9q^2)rs^3+2p(p-5q)rs^2t\\
&& \quad -2p(p-5q)rst^2-(p^2-q^2)rt^3-(p^2-9q^2)s^3t\\
&& \quad +(p^2-q^2)st^3,\\
d &=& -2(r-s)(s-t)(t-r)(s+t)prst,\\
e &=& -(p-5q)^2r^4s^2+(p-5q)^2r^4t^2+2p(p-5q)r^3s^3\\
&& \quad +2p(p-5q)r^3s^2t-2p(p-5q)r^3st^2-2p(p-5q)r^3t^3\\
&& \quad -(p^2-9q^2)r^2s^4-2p(p-5q)r^2s^3t+2p(p-5q)r^2st^3\\
&& \quad +(p^2-q^2)r^2t^4+(p^2-9q^2)s^4t^2-(p^2-q^2)s^2t^4,\\
f &=& rst\{(p-5q)^2r^4s-(p-5q)^2r^4t-2p(p-5q)r^3s^2\\
&& \quad +2p(p-5q)r^3t^2+2p(p-5q)r^2s^2t-2p(p-5q)r^2st^2\\
&& \quad +(p^2-9q^2)rs^4-(p^2-q^2)rt^4-(p^2-9q^2)s^4t\\
&& \quad +(p^2-q^2)st^4\}.
\end{array} \tag{9}
$$

Substituting these values of a, b, c, d, e, f, in the last equation given by (8), we get the following quadratic equation in p and q:

$$
\begin{array}{l}
(r-1)(s-1)(r-s)(r-t)(s-t)(rs-t)(rs+r-s-t)p^2\\
\quad +10r(r-1)(r-s)(r-t)(s-t)(rs-t)(s^2+1)pq\\
\quad +(s-1)\{(25(s+1)(s-t)r^5s-25(s^2-t^2)(s^2+s+1)r^4\\
\qquad +25(s+1)(s^2+1)(s-t)r^3t+(9s^4-t^4)r^2\\
\qquad -(s+1)(9s^3-t^3)rt+(9s^2-t^2)st^2\}q^2=0.
\end{array} \tag{10}
$$

We take $t=rs+r-s$ when the coefficient of p^2 vanishes, and we then solve equation (10) to get

$$
\begin{array}{l}
p=(s-1)\{(12s^2-s+12)(s+1)^2r^4-(s+1)(23s^2-4s+23)r^3s\\
\qquad -3(s+1)^2r^2s^2+2(s+1)rs^3+4s^4\},\\
q=5rs(r-1)(r-s)(rs+r-2s)(s^2+1).
\end{array} \tag{11}
$$

With p, q being defined by (11) and $t=rs+r-s$, the coefficients of the conic (1) are given by (9) in terms of parameters r and s, and all the six conditions (8) are satisfied. We have thus obtained a parametrised family of conics on which there are six points $(x_j,\, y_j)$ such that the six products x_jy_j are in arithmetic progression.

We give two numerical examples. Taking $r = 4,\ s = 3$ we get the ellipse

$$\begin{aligned} x^2 + 306084xy + 31920349248y^2 - 19589376x & \\ -3541188192768y + 101175664599360 &= 0, \end{aligned}$$

on which there are the six points $(x_1,\, y_1) = (2088672,\, 39),\ (x_2,\, y_2) = (1674504,\, 52), (x_3,\, y_3) = (7724184,\, 12),\ (x_4,\, y_4) = (8192184,\, 12),\ (x_5,\, y_5) = (1998504,\, 52), (x_6,\, y_6) = (8426016,\, 13)$ such that the products x_jy_j, namely, $81458208,\ 87074208, 92690208,\ 98306208,\ 103922208, 109538208$ are in arithmetic progression.

As a second example, when we take $r = 2,\ s = 3$, we get the hyperbola

$$5x^2 + 80520xy + 229170816y^2 - 1288320x - 8878709376y + 77394458880 = 0,$$

on which there are the six points $(x_1,\, y_1) = (38208,\, 15),\ (x_2,\, y_2) = (53712, 10), (x_3,\, y_4) = (83520, 6),\ (x_4,\, y_4) = (77520, 6),\ (x_5,\, y_5) = (42912,\, 10), (x_6,\, y_6) = (78624,\, 5)$ such that the products x_jy_j, namely, $573120,\ 537120,\ 501120, 465120, 429120,\ 393120$ are in arithmetic progression.

By assigning different numerical values to the parameters $r,\ s$, we can obtain infinitely many ellipses and infinitely many hyperbolas on which there exist six points with the desired property. Efforts to determine suitable values of the parameters $r,\ s$ to get a parabola were futile. We could, of course, solve only the first five of the equations (8) and choose the parameters to get a parabola with five points $(x_j,\, y_j)$ such the five products x_jy_j are in arithmetic progression. This solution is, however, more cumbersome than the solution given in Section 2, and is accordingly omitted.

4 Concluding remarks

The examples given in Section 2 show that there exist parabolas on which there are five points $(x_j,\, y_j)$ such that the products x_jy_j are in arithmetic progression. Similarly, in Section 3 we have shown that there exist ellipses and hyperbolas on which there are six points $(x_j,\, y_j)$ such that the products x_jy_j are in arithmetic progression. These examples disprove Brown's conjecture.

It remains an open problem whether there exist parabolas on which there are six points or ellipses / hyperbolas on which there are seven points such that the areas of the corresponding rectangles are in arithmetic progression.

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