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Fibonacci number sums as prime indicators

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Abstract: Sums of the first p Fibonacci numbers, S_p, are shown to be related to K in $F_p = Kp \pm 1$, which is itself a useful indicator of primality for F_p. Digit sums of K, S_p, sums of F_p^2 and Simson's identity were compared.

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1 Introduction

The sum of the first n Fibonacci numbers is

$$\sum_{i=1}^{n} F_n = F_{n+2} - 1 \tag{1.1}$$

which, when $n = p$, becomes

$$S_p = F_{p+2} - 1. \tag{1.2}$$

Following on recent work [3–7], we compare this sum for the primes $3 \le p \le 113$ using the method of digit sums [2, 7]. The digit sum in base b, of the digits of the number n, is often represented by $s_b(n)$ [1], and satisfies the congruence:

$$n \equiv s_b(n) \pmod{b-1}. \tag{1.3}$$

When $b = 10$, this congruence is the basis of high school techniques such as casting out nines and of divisibility tests such as those for 3 and 9.

2 Digit sums of S_p

The sums of Fibonacci numbers and the corresponding digit sums (Tables 1 and 2) show distinctions between primes and composites (Table 2).F_{97}, if a prime, does not conform so that it was checked with the factor ($k\backslash Kp \pm 1$) technique [3,4,6] which showed that it was indeed a composite, namely

$$F_{97} = 83621143489848422977$$
$$= 193 \times 389 \times 1113805073322701$$

with

$$193 = 2 \times 97 - 1$$

and

$$389 = 4 \times 97 + 1.$$

However, either F_7 or F_{37} do not conform, both having a digit sum of 6. F_{109}, if conforming, is a composite, and F_{113} is a prime. These results are generally in accord with the K results, from [6]

$$F_p = Kp \pm 1 \qquad (2.1)$$

The K values appear to be more reliable.

p	**S_p**	**Digit sum**	**Parity**	**p**	**S_p**	**Digit sum**	**Parity**
3	4	4	p	**59**	2504730781960	7	c
7	33	6	p	**61**	6557470319841	8	p
11	332	7	p	**67**	117669030460993	1	c
13	609	6	p	**71**	806515533049392	9	p
17	4180	4	p	**73**	2111485077978049	1	p
19	10945	1	c	**79**	37889062373143905	6	c
23	75024	9	p	**83**	259695496911122584	7	p
29	1346268	3	p	**89**	4660046610375530308	4	c
31	3524577	6	c	**97**	21892295834555169025	1	c
37	63245985	6	c	**101**	150052053620689608327 6	3	p
41	433494436	4	c	**103**	3928413764606871165729	6	p
43	1134903169	1	p	**107**	26925748508234281076008	7	c
47	7778742048	9	p	**109**	70492524767089125814113	6	c
53	139583862444	3	c	**113**	483162952612010163284884	4	p

Table 1. Digit sums of first p Fibonacci numbers and parity

p*	**Primes**	**Composites**
1	3,7,8,9	4,6
3	1,4,6,7,9	3
7	4,6,9	1,6,7
9	3	1,4,6,7

Table 2. Digit sum for first p Fibonacci Numbers: [p* = right-end-digit]

p*	**F_{p+1}**	**F_{p-1}**
1		✓
3	✓	
7	✓	
9		✓

Table 3. Divisibility of neighbours of F_p by p

3 Relationship of S_p and K in $F_p = K_p \pm 1$

One of the direct neighbours of F_p is divisible by p according to p^* in a neat pattern (Table 3) [3].

1.1 $p^* = 3, 7$

From (1.2) and the Fibonacci recurrence relation

$$S_p + 1 = F_{p+1} + Kp - 1 \tag{3.1}$$

$$F_{p+2} = F_{p+1} + F_p \tag{3.2}$$

so that

$$K = \frac{1}{p}\left(S_p + 2 - F_{p+1}\right). \tag{3.3}$$

Both (S_p+2) and F_{p+1} are divisible by p; K values (Table 4) from Equation (3.3) agree with those from Equation (2.1).

p	Parity	K	p	Parity	K
3	p	1	**53**	c	1005967758
7	p	2	**67**	c	670829406162
13	p	18	**73**	p	11048157986978
17	p	94	**83**	p	1195118711985006
23	p	1246	**97**	c	862073644225241474
37	c	652914	**103**	p	14568160545698020226
43	p	10081266	**107**	c	96118885585174929102
47	p	63217342	**113**	p	16332019981684334481118

Table 4. K values from Equation (3.3)

1.2 $p^* = 1, 9$

Since from repeating the Fibonacci recurrence relation

$$\begin{aligned} F_{p+2} &= 2F_p + F_{p-1} \\ &= 2(Kp+1) + F_{p-1} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} S_p &= 2(Kp+1) + F_{p-1} - 1 \\ &= 2Kp + 1 + F_{p-1} \end{aligned} \tag{3.5}$$

then

$$K = \frac{1}{2p}\left(S_p - 1 - F_{p-1}\right). \tag{3.6}$$

As for Section (3.1), K values agree with previous values [6,7]. The sums of digits of K from Tables 4 and 5 show clear distinctions for primes and composites (Table 6).

p	Parity	*K*	*p*	Parity	*K*
11	p	8	**61**	p	41061160360
19	c	220	**71**	p	4338894664368
29	p	17732	**79**	c	183194101578180
31	c	43428	**89**	c	19999768719154092
41	c	4038540	**101**	p	5674731128849674100
59	c	16215627560	**109**	c	494050431343748276624

Table 5. *K* values from Equation (3.6)

*p**	Primes	Composites
1	1,2,8,9	3,6
3	1,2,4,6,8,9	3
7	1,2,4	6,8,9
9	2	3,4,5,6

Table 6. Digit sums of *K*

4 Sums of Fibonacci squares

The digit sums of Fibonacci squares were calculated from

$$F_p \times F_{p+1} = \sum_{i=1}^{p} F_p^2. \tag{4.1}$$

The digit sums generally show distinctions between primes and composites (Tables 7, 8) but not as reliably as *K* or S_p. The digit sums are simply obtained from the digit sums of F_p times the digit sums of F_{p+1}.

p	Parity	F_p	F_{p+1}	F_pF_{p+1}	*p*	Parity	F_p	F_{p+1}	F_pF_{p+1}
3	p	2	3	6	**59**	c	8	9	9
7	p	4	3	3	**61**	p	8	8	1
11	p	8	9	9	**67**	c	5	6	3
13	p	8	8	1	**71**	p	1	9	9
17	p	4	1	4	**73**	p	1	1	1
19	c	5	6	3	**79**	c	4	3	3
23	p	1	9	9	**83**	p	8	9	9
29	p	5	8	4	**89**	c	4	1	4
31	c	4	3	3	**97**	c	1	1	1
37	c	8	8	1	**101**	p	5	8	4
41	c	4	1	4	**103**	p	4	5	2
43	p	5	6	3	**107**	c	8	8	1
47	p	1	9	9	**109**	c	8	8	1
53	c	5	8	4	**113**	p	4	1	4

Table 7. Digit sums of Fibonacci squares

p^*	Primes	Composites
1	1,4,9	3,4
3	1,2,3,4,6,9	4
7	3,4,9	1,3
9	4	1,3,4,9

Table 8. Digit sums

The digit sums 3 and 4 are clearly indeterminate for this index.

5 Simson's Identity

An approximation to this identity leads to the Golden Ratio [5], namely

$$F_{n+1}F_{n-1} \approx F_n^2$$

whereas the precise form is

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n \tag{5.1}$$

and

$$F_{n+2}F_{n-2} = F_n^2 + (-1)^{n-1} \tag{5.2}$$

and the digit sums of these quantities show that the digit sum of F_p^2 is either 1 or 7. (Compare the perfect numbers which always have a digit sum of 1 [7].) The composite F_p commonly have a digit sum of 7 when p^* = 1, 3, or 9, but digit sum of 1 when p^* = 7.

p	Parity of F_p	F_{p-1}x F_{p+1}	F_{p-2}x F_{p+2}	F_p^2	p	Parity of F_p	F_{p-1}x F_{p+1}	F_{p-2}x F_{p+2}	F_p^2
11	p	9	2	1	**7**	p	6	8	7
31	c	6	8	7	**17**	p	6	8	7
41	c	6	8	7	**37**	c	9	2	1
61	p	9	2	1	**47**	p	9	2	1
71	p	9	2	1	**67**	c	6	8	7
101	p	6	8	7	**97**	c	9	2	1
13	p	9	2	1	**107**	c	9	2	1
23	p	9	2	1					
43	p	6	8	7	**19**	c	6	8	7
53	c	6	8	7	**29**	p	6	8	7
73	p	9	2	1	**59**	c	9	2	1
83	p	9	2	1	**79**	c	6	8	7
103	p	6	8	7	**89**	c	6	8	7
113	p	6	8	7	**109**	c	9	2	1

Table 9. Digit sums for F_p^2

6 Concluding comments

While the results are interesting because of the links, other indicators such as K or S_p are more useful for primality tests [6, 7]. Furthermore, digit sums provide patterns for further exploration in both pure [8] and applied mathematics [2] for university student projects at all levels, as do right-end-digits as integers (modulo 10) [9].

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