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On some Pascal's like triangles. Part 6

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To my friend since kindergarden
Evgeni Dimitrov for his 60-th Anniversary!

Abstract: A series of Pascal's like triangles with different forms are described and some of their propertiesa are given.
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1 Introduction

In a series of the five papers [1, 2, 3, 4, 5], we discussed a new type of Pascal's like triangles. Triangles from the present form, but not with the present sense, are described in different publications, e.g. [6, 7, 8, 10].

2 Main results

Now, we complicate the form of the Pascal's like triangles from [1]. Let us start with the following examples:

$$\mathbf{E}_1 : \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{0} & & & & & & \\
 & & & & & 1 & \mathbf{2} & 3 & & & & & \\
 & & & & 1 & 2 & \mathbf{6} & 6 & 3 & & & & \\
 & & & 1 & 2 & 4 & \mathbf{14} & 12 & 6 & 3 & & & \\
 & & 1 & 2 & 4 & 8 & \mathbf{30} & 24 & 12 & 6 & 3 & & \\
 & 1 & 2 & 4 & 8 & 16 & \mathbf{62} & 48 & 24 & 12 & 6 & 3 & \\
1 & 2 & 4 & 8 & 16 & 32 & \mathbf{126} & 96 & 48 & 24 & 12 & 6 & 3 \\
 & & & & . & & . & & . & & & &
\end{array}$$

$\mathbf{E}_2$:

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{0} & & & & & & \\
 & & & & & 1 & \mathbf{3} & 5 & & & & & \\
 & & & & 1 & 2 & \mathbf{9} & 10 & 5 & & & & \\
 & & & 1 & 2 & 4 & \mathbf{21} & 20 & 10 & 5 & & & \\
 & & 1 & 2 & 4 & 8 & \mathbf{45} & 40 & 20 & 10 & 5 & & \\
 & 1 & 2 & 4 & 8 & 16 & \mathbf{93} & 80 & 40 & 20 & 10 & 5 & \\
1 & 2 & 4 & 8 & 16 & 32 & \mathbf{189} & 160 & 80 & 40 & 20 & 10 & 5 \\
 & & & & . & & . & & . & & & &
\end{array}$$

$\mathbf{E}_3$:

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{0} & & & & & & \\
 & & & & & 3 & \mathbf{4} & 5 & & & & & \\
 & & & & 3 & 6 & \mathbf{12} & 10 & 5 & & & & \\
 & & & 3 & 6 & 12 & \mathbf{28} & 20 & 10 & 5 & & & \\
 & & 3 & 6 & 12 & 24 & \mathbf{60} & 40 & 20 & 10 & 5 & & \\
 & 3 & 6 & 12 & 24 & 48 & \mathbf{124} & 80 & 40 & 20 & 10 & 5 & \\
3 & 6 & 12 & 24 & 48 & 96 & \mathbf{252} & 160 & 80 & 40 & 20 & 10 & 5 \\
 & & & & . & & . & & . & & & &
\end{array}$$

In [1] we discussed infinite triangles with the form

$$\begin{array}{ccccccc}
 & & & a_{1,1} & & & \\
 & & a_{2,1} & a_{2,2} & a_{2,3} & & \\
 & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\
 & . & & . & & . &
\end{array}$$

where $a_{i,1} = a_{i,2i-1}$ are arbitrary real (complex) numbers and for every natural number $i \geq 1$ and
1. for every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} = a_{i,j-1} + a_{i-1,j-1};$$

2. for every natural number j for which $i \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i,j+1} + a_{i-1,j-1}.$$

Now, we complicate the above scheme to the following form: $a_{i,1}$ and $a_{i,2i-1}$ are arbitrary real (complex) numbers (i.e., without the above condition for them to be equal) and for every natural number $i \geq 1$ and
1. for every natural number j for which $2 \leq j \leq i-1$ it will be valid:

$$a_{i,j} = a_{i,j-1} + a_{i-1,j-1};$$

2. for every natural number j for which $i+1 \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i,j+1} + a_{i-1,j-1};$$

3. for $i \geq 2$:

$$a_{i,i} = a_{i-1,i-1} + \frac{a_{i,i-1} + a_{i,i+1}}{2}.$$

Let each of the sequences $\{a_{i,1}\}_{i\geq 1}$ and $\{a_{2i-1,1}\}_{i\geq 1}$ be called a *generating sequence* and let each of the sequences $\{a_{i,i}\}_{i\geq 1}$ be called a *generated sequence.*

We can prove, e.g., by induction

Lemma: For every two real (complex) numbers $p \geq 1$ and $q \geq 0$:

$$\begin{array}{ccccccccc}
 & & & & \mathbf{0} & & & & \\
 & & & 2p-1 & \mathbf{p+q} & 2q+1 & & & \\
 & & 2p-1 & 4p-2 & \mathbf{3}(\mathbf{p+q}) & 4q+2 & 2q+1 & & \\
 & 2p-1 & 4p-2 & 8p-4 & \mathbf{7}(\mathbf{p+q}) & 8q+4 & 4q+2 & 2q+1 & \\
2p-1 & 4p-2 & 8p-4 & 16p-8 & \mathbf{15}(\mathbf{p+q}) & 16q+8 & 8q+4 & 4q+2 & 2q+1 \\
 & & . & & . & & . & &
\end{array}$$

The i-th member of the generated sequence has the form

$$a_{i,i} = (2^{i-1} - 1).(p + q).$$

Hence, the i-th members of the generated sequences $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ are, respectively, $2(2^{i-1} - 1), 3(2^{i-1} - 1)$ and $4(2^{i-1} - 1)$

In [1], the following triangle was discussed:

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 1 & \mathbf{2} & 1 & & & & & \\
 & & & & 1 & 2 & \mathbf{4} & 2 & 1 & & & & \\
 & & & 1 & 2 & 4 & \mathbf{8} & 4 & 2 & 1 & & & \\
 & & 1 & 2 & 4 & 8 & \mathbf{16} & 8 & 4 & 2 & 1 & & \\
 & 1 & 2 & 4 & 8 & 16 & \mathbf{32} & 16 & 8 & 4 & 2 & 1 & \\
1 & 2 & 4 & 8 & 16 & 32 & \mathbf{64} & 32 & 16 & 8 & 4 & 2 & 1 \\
 & & & & . & & . & & . & & & &
\end{array}$$

Therefore, the i-th member of its generated sequence is 2^{i-1}. Now, changing the forms of the generating sequences, we obtain the following new Pascal's like triangles.

$$\mathbf{E}_4 : \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 1 & \mathbf{3} & 3 & & & & & \\
 & & & & 1 & 2 & \mathbf{6} & 4 & 1 & & & & \\
 & & & 1 & 2 & 4 & \mathbf{12} & 8 & 4 & 3 & & & \\
 & & 1 & 2 & 4 & 8 & \mathbf{24} & 16 & 8 & 4 & 1 & & \\
 & 1 & 2 & 4 & 8 & 16 & \mathbf{48} & 32 & 16 & 8 & 4 & 3 & \\
1 & 2 & 4 & 8 & 16 & 32 & \mathbf{96} & 64 & 32 & 16 & 8 & 4 & 1 \\
 & & & & . & & . & & . & & & &
\end{array}$$

$$\mathbf{E}_5: \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 2 & \mathbf{4} & 4 & & & & & \\
 & & & & 1 & 3 & \mathbf{8} & 5 & 1 & & & & \\
 & & & 2 & 3 & 6 & \mathbf{16} & 10 & 5 & 4 & & & \\
 & & 1 & 3 & 6 & 12 & \mathbf{32} & 20 & 10 & 5 & 1 & & \\
 & 2 & 3 & 6 & 12 & 24 & \mathbf{64} & 40 & 20 & 10 & 5 & 4 & \\
1 & 3 & 6 & 12 & 24 & 48 & \mathbf{128} & 80 & 40 & 20 & 10 & 5 & 1 \\
 & & & & . & & . & & . & & & &
\end{array}$$

$$\mathbf{E}_6: \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 2 & \mathbf{5} & 6 & & & & & \\
 & & & & 1 & 3 & \mathbf{10} & 7 & 1 & & & & \\
 & & & 2 & 3 & 6 & \mathbf{20} & 14 & 7 & 6 & & & \\
 & & 1 & 3 & 6 & 12 & \mathbf{40} & 28 & 14 & 7 & 1 & & \\
 & 2 & 3 & 6 & 12 & 24 & \mathbf{80} & 56 & 28 & 14 & 7 & 6 & \\
1 & 3 & 6 & 12 & 24 & 48 & \mathbf{160} & 112 & 56 & 28 & 14 & 7 & 1 \\
 & & & & . & & . & & . & & & &
\end{array}$$

$$\mathbf{E}_7: \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 2 & \mathbf{6} & 8 & & & & & \\
 & & & & 1 & 3 & \mathbf{12} & 9 & 1 & & & & \\
 & & & 2 & 3 & 6 & \mathbf{24} & 18 & 9 & 8 & & & \\
 & & 1 & 3 & 6 & 12 & \mathbf{48} & 36 & 18 & 9 & 1 & & \\
 & 2 & 3 & 6 & 12 & 24 & \mathbf{96} & 72 & 36 & 18 & 9 & 8 & \\
1 & 3 & 6 & 12 & 24 & 48 & \mathbf{192} & 144 & 72 & 36 & 18 & 9 & 1 \\
 & & & & . & & . & & . & & & &
\end{array}$$

More general, we can consruct the triangle

$$\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & 2 & \mathbf{p+2} & 2p & & \\
 & 1 & 3 & \mathbf{2p+4} & 2p+1 & 1 & \\
2 & 3 & 6 & \mathbf{4k+8} & 4p+2 & 2p+1 & 2p \\
 & . & & . & & . &
\end{array}$$

Therefore, the i-th member of the generated sequence has the form

$$a_{i,i} = (p+1)2^{i-1}.$$

Obviously, the i-th member of the generated sequences $\mathbf{E}_4$, $\mathbf{E}_5$, $\mathbf{E}_6$ and $\mathbf{E}_7$ are, respectively, $3.2^{i-1}, 4.2^{i-1}, 5.2^{i-1}$ and 6.2^{i-1}.

On the other hand, we can see that we can construct a triangle with other generating sequences, which has the same generated sequence as, e.g., the last triangle:

$$\mathbf{E}_8: \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 4 & \mathbf{6} & 6 & & & & & \\
 & & & & 1 & 5 & \mathbf{12} & 7 & 1 & & & & \\
 & & & 4 & 5 & 10 & \mathbf{24} & 14 & 7 & 6 & & & \\
 & & 1 & 5 & 10 & 20 & \mathbf{48} & 28 & 14 & 7 & 1 & & \\
 & 4 & 5 & 10 & 20 & 40 & \mathbf{96} & 56 & 28 & 14 & 7 & 6 & \\
1 & 5 & 10 & 20 & 40 & 80 & \mathbf{192} & 112 & 56 & 28 & 14 & 7 & 1 \\
 & & & & . & & . & & . & & & &
\end{array}$$

Hence, we can consruct the more general form of a triangle

$$\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & 2p & \mathbf{p+q+1} & 2q & & \\
 & 1 & 2p+1 & \mathbf{2}(\mathbf{p+q+1}) & 2q+1 & 1 & \\
2p & 2p+1 & 4p+2 & \mathbf{4}(\mathbf{p+q+1}) & 4q+2 & 2q+1 & 2q \\
 & . & & . & & . &
\end{array}$$

Therefore, the i-th member of the generated sequence of this triangle has the form

$$a_{i,i} = (p+q+1)2^{i-1}.$$

Of course, we can construct a lot of other forms of Pascal's like triangles. For example:

$$\mathbf{E}_9: \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{0} & & & & & & \\
 & & & & & 2 & \mathbf{3} & 4 & & & & & \\
 & & & & 4 & 6 & \mathbf{9} & 6 & 2 & & & & \\
 & & & 2 & 6 & 12 & \mathbf{21} & 12 & 6 & 4 & & & \\
 & & 4 & 6 & 12 & 24 & \mathbf{45} & 24 & 12 & 6 & 2 & & \\
 & 2 & 6 & 12 & 24 & 48 & \mathbf{93} & 48 & 24 & 12 & 6 & 4 & \\
4 & 6 & 12 & 24 & 48 & 96 & \mathbf{189} & 96 & 48 & 24 & 12 & 6 & 2 \\
 & & & & . & & . & & . & & & &
\end{array}$$

Its i-th member of the generated sequence has the form

$$a_{i,i} = 6(2^{i-1} - 1) + 3.$$

3 Conclusion

In Conclusion, we study three Pascal's like triangles with special generating sequences: Fibonacci $(f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, ...)$, Lucas $(2, 1, 3, 4, 7, 11, ...)$, Jacobstal $(1, 1, 3, 5, 11, 21, ...)$ and Jacobstal–Lucas $(1, 5, 7, 17, 31, 65, ...)$ sequences (cf. [1, 7, 9, 11]).

In the first case, the left generating sequence is the Fibonacci sequence and the right generating sequence is the Lucas sequence, but starting with its second member:

$$\mathbf{E}_{10}: \quad \begin{array}{c}
\mathbf{1} \\
1 \quad \mathbf{3} \quad 3 \\
2 \quad 3 \quad \mathbf{8} \quad 7 \quad 4 \\
3 \quad 5 \quad 8 \quad \mathbf{21} \quad 18 \quad 11 \quad 7 \\
5 \quad 8 \quad 13 \quad 21 \quad \mathbf{55} \quad 47 \quad 29 \quad 18 \quad 11 \\
8 \quad 13 \quad 21 \quad 34 \quad 55 \quad \mathbf{144} \quad 123 \quad 76 \quad 47 \quad 29 \quad 18 \\
13 \quad 21 \quad 34 \quad 55 \quad 89 \quad 144 \quad \mathbf{366} \quad 322 \quad 199 \quad 123 \quad 76 \quad 47 \quad 29 \\
. \qquad . \qquad .
\end{array}$$

Therefore, the generated sequence coincides with sequence $\{f_{2k}\}_{k\geq 1}$.

In the second case, the left generating sequence is the Fibonacci sequence, but starting with its third member and the right generating sequence is the Lucas sequence:

$$\mathbf{E}_{11}: \quad \begin{array}{c}
\mathbf{2} \\
3 \quad \mathbf{4} \quad 1 \\
5 \quad 8 \quad \mathbf{10} \quad 4 \quad 3 \\
8 \quad 13 \quad 21 \quad \mathbf{26} \quad 11 \quad 7 \quad 4 \\
13 \quad 21 \quad 34 \quad 55 \quad \mathbf{68} \quad 29 \quad 18 \quad 11 \quad 7 \\
21 \quad 34 \quad 55 \quad 89 \quad 144 \quad \mathbf{178} \quad 76 \quad 47 \quad 29 \quad 18 \quad 11 \\
34 \quad 55 \quad 89 \quad 144 \quad 233 \quad 377 \quad \mathbf{466} \quad 199 \quad 123 \quad 76 \quad 47 \quad 29 \quad 18 \\
. \qquad . \qquad .
\end{array}$$

Therefore, the generated sequence coincides with sequence $\{2f_{2k+1}\}_{k\geq 1}$.

In the third case, the left generating sequence is the Jacobstal sequence and the right generating sequence is the Jacobstal–Lucas sequence:

$$\mathbf{E}_{12}: \quad \begin{array}{c}
\mathbf{1} \\
1 \quad \mathbf{4} \quad 5 \\
3 \quad 4 \quad \mathbf{12} \quad 12 \quad 7 \\
5 \quad 8 \quad 12 \quad \mathbf{36} \quad 36 \quad 24 \quad 17 \\
11 \quad 16 \quad 24 \quad 36 \quad \mathbf{108} \quad 108 \quad 72 \quad 48 \quad 31 \\
21 \quad 32 \quad 48 \quad 72 \quad 108 \quad \mathbf{324} \quad 324 \quad 216 \quad 144 \quad 96 \quad 65 \\
. \qquad . \qquad .
\end{array}$$

Therefore, the i-th member of the generated sequence has the form

$$a_{i,i} = 4.3^{i-2}$$

for $i \geq 2$.

In a next paper, similarly to [4], three-dimensional analogues of the described here Pascal's like triangles will be discussed.

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