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On two new means of two variables II

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Dedicated to the 65th Anniversary
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Abstract: Here authors establish the inequalities for two means X and Y studied in [11], and give the series expansion of these means.
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1 Introduction

The study of the classical means such as Logarithmic mean, Identric mean, Arithmetic mean, Geometric mean and Seiffert mean has been topic of the intensive research in past few decades. For the inequalities and the applications of these means we refer to reader to see [2–4, 6, 7, 12, 13, 17, 18] and the references therein.

Recently, Sándor [11] discovered two means X and Y, defined as

$$X = X(a,b) = Ae^{G/P-1}, \quad Y = Y(a,b) = Ae^{L/A-1},$$

for two distinct real numbers a and b, where

$$A = (a,b) = (a+b)/2, \quad G = G(a,b) = \sqrt{ab}$$

$$L = L(a,b) = \frac{a-b}{\log(a) - \log(y)}, \quad a \neq b,$$
$$P = P(a,b) = \frac{a-b}{2\arcsin\left(\frac{a-b}{a+b}\right)}, \quad a \neq b,$$

are Arithmetic, Geometric, Logarithmic and Seiffert [19] means. The Harmonic and Identric means of two real numbers a and b are respectively defined as

$$H = H(a,b) = \frac{ab}{a+b}, \quad I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}.$$

For the properties of X and Y means, and their relations with other means such as A, G, I, L, we refer to reader to see [11].

One of the main results of the paper reads as follows:

Theorem A. For $a > b > 0$, we have

(1) $\frac{1}{e}(G+H) < Y < \frac{1}{2}(G+H)$,

(2) $G^2 I < IY < IG < L^2$,

(3) $\frac{G-Y}{A-L} < \frac{Y+G}{2A} < \frac{3G+H}{4A} < 1$,

(4) $L < \frac{2G+A}{3} < X < L(X,A) < P < \frac{2A+G}{3} < I$,

(5) $2\left(1 - \frac{A}{P}\right) < \log\left(\frac{X}{A}\right) < \left(\frac{P}{A}\right)^2.$

The binomial coefficient $\binom{2n}{n}$ will be denoted for simplicity by $(2n, n)$.

Theorem B. One has the representations

(1) $\log\left(\frac{X}{A}\right) = -\frac{G}{A}\sum_{n=0}^{\infty} a_n t^n$,
where
$$a_n = \frac{2n}{4^n(2n+1)}, \quad (2n,n) = \frac{(2n)!}{(n!)^2}, \quad t = \left(\frac{b-a}{b+a}\right)^2,$$

(2) $\log\left(\frac{Y}{G}\right) = \sum_{n=1}^{\infty} b_n t^n$,
where
$$b_1 = -\frac{1}{3},\ b_2 = -\frac{4}{45}, \cdots, b_n = -\left(\frac{1}{2n+1} + b_{1/(2n-1)} + b_{2/(2n-3)} + \cdots + b_{(n-1)/3}\right).$$

2 Preliminaries

We give the following lemmas, which will be used in the proofs of our mains results.

Lemma 1. For $x > 1$, we have

(1) $\log(x) < \sqrt{\dfrac{9x^2-1}{6x}} - 1$,

(2) $\log\left(\dfrac{2(3x-1)}{3x+1}\right) < \dfrac{\log(x)}{1+\log(x)}$.

Proof. Let

$$f(x) = 1 + \log x - \sqrt{\frac{9x^2-1}{6x}},$$

for $x > 1$. One has

$$\begin{aligned} xf'(x) &= 1 - \frac{1+9x^2}{2\sqrt{6x(9x^2-1)}} = \\ &= -\frac{P(x)}{2\sqrt{6x(9x^2-1)}(1+9x^2+2\sqrt{6x(9x^2-1)})}, \end{aligned}$$

where

$$P(x) = 81x^4 - 216x^3 + 18x^2 + 24x + 1.$$

Again, we have

$$P'(x) = 324x^3 - 648x^2 + 36x + 24, \quad P''(x) = 972x^2 - 1296x + 36.$$

Since $81x^4 - 216x^3 = x^3(81x - 216) > 0$ for $x \geq 3$, we get $P(x) > 0$ for $x \geq 3$, so $f'(x) < 0$ for $x \geq 3$. Thus $f(x) < f(3) < 0$. Therefore, we have to consider x in $(1,3)$. The equation $P''(x) = 0$ has a root $x_1 = 1.3$ in $(1,2)$, and from the graph of the parabola it follows that $P''(x) < 0$ for $x \in (1, x_1)$ and $P''(x) > 0$ for $x \in (x_1, 3)$. By $P'(1) < 0$ and $P'(2) > 0$ it follows that $P'(x) = 0$ has a root x_2 in $(1,2)$. Since $P'(x)$ is strictly decreasing in $(1, x_1)$, the root x_2 is in fact in $P'(x_1), 2)$. Since $P'(x)$ is strictly increasing in $(x_1, 2)$, the root x_2 is unique. Therefore, we get that $P'(x) < 0$ for x in $(1, x_2)$ and $P'(x) > 0$ in $(x_2, 3)$. This implies that $P(x)$ has a unique root x_3 in $(x_2, 3)$.

Now the following can be written: For $x \in (1, x_2)$ one has $P(x) < P(1) < 0$ (as $P(x)$ is strictly decreasing here). For $x \in (x_2, x_3)$ one has $P(x) < P(x_3) = 0$ (as $P(x)$ is strictly increasing here). For $x \in (x_3, 3)$ one has $P(x) > P(x_3) = 0$ (as $P(x)$ is strictly increasing here). The above imply that $P(x) < 0$ for $x \in (1, x_3)$ and $P(x) > 0$ for $x \in (x_3, 3)$. Therefore, $f'(x) > 0$ in the first interval, while it is < 0 in the second one. In other words, the point x_3 is a maximum point of $f(x)$ in the interval $(1,3)$. This means that $f(x) \leq f(x_3)$. An easy computation shows that x_3 is approximately 2.6. Since $f(2.6) < 0$, the inequality is verified in all cases.

For the proof of part (2), write

$$g(x) = \log\left(\frac{2(3x-1)}{3x+1}\right) - \frac{\log(x)}{1+\log(x)},$$

for $x > 1$. We get

$$g'(x) = \frac{6}{9x^2 - 1} - \frac{1}{x(1 + \log(x))^2},$$

which is negative by part (1), and $g(1) = 0$. This implies the proof of part (2). $\square$

Lemma 2. For $x > 0$, the following inequalities hold true

(1) $1 + \log\left(\frac{1 + 2\cosh(x)}{3}\right) < \frac{x}{\tanh(x)} < \log(1 + 2\cosh(x))$,

(2) $\frac{\tanh(x)}{x} < 1 - \log\left(\frac{2\cosh(x)}{1 + \cosh(x)}\right)$.

Proof. For $x > 0$, let

$$m(x) = 1 + \log\left(\frac{1 + 2\cosh(x)}{3}\right) - \frac{x}{\tanh(x)}.$$

Differentiating m with respect to x, and applying the inequality

$$\cosh(x)^{1/3} < \frac{\sinh(x)}{x},\ x > 0,$$

we get

$$\begin{aligned} m'(x) &= \frac{x}{\sinh(x)^2} + \frac{\cosh(x)}{\sinh(x)} + \frac{2 + \sinh(x)}{1 + 2\cosh(x)} \\ &= \frac{x + +2x\cosh(x) - (2 + \cosh(x))\sinh(x)}{\sinh(x)^2(1 + 2\cosh(x))} \\ &= \frac{x(2 + \cosh(x))}{\sinh(x)^2(1 + 2\cosh(x))}\left(\frac{1 + 2\cosh(x)}{2 + \cosh(x)} - \frac{\sinh(x)}{x}\right) \\ &< \frac{x(2 + \cosh(x))}{\sinh(x)^2(1 + 2\cosh(x))}\left(\frac{1 + 2\cosh(x)}{2 + \cosh(x)} - \cosh(x)^{1/3}\right) < 0, \end{aligned}$$

the last inequality follows because the function

$$m_1(z) = \log\left(\frac{1 + 2z}{2 + z}\right),\ z > 1$$

has derivative

$$m_1'(z) = -\frac{2(z - 1)^2}{3z(2 + z)(1 + 2z)} < 0,$$

and $w_1(1) = 0$. Since, $m(x)$ is strictly decreasing and we get

$$\lim_{x \to 0} m(x) = 0 > m(x) > \lim_{x \to \infty} m(x) = 1 - \log(3),$$

this implies the proof.

For (2), Replacing x by $(1+2\cosh(x))/3$ in Lemma 1(2) we get

$$\begin{aligned} f(x) &= \log\left(\frac{2\cosh(x)}{1+\cosh(x)}\right)+\frac{\tanh(x)}{x}-1 \\ &< \frac{\log(1+2\cosh(x))/3}{1+\log((1+2\cosh(x))/3)}+\frac{\tanh(x)}{x}-1 \\ &= \frac{\tanh(x)}{x}-\frac{1}{1+\log((1+\cosh(x))/3)}, \end{aligned}$$

which is negative by the first inequality of part (1). □

Lemma 3. For $a>b>0$, one can find an x in $(0,\pi/2)$ and an y in $(0,\infty)$ such that

(1) $a=(1+\sin(x))A$ and $b=(1-\sin(x))A$,

(2) $a=e^yG$ and $b=e^{-y}G$,

where $A=A(a,b)$ and $G=G(a,b)$.

Proof. Proof. It is immediate that $x=\arcsin((a-b)/(a+b))$, so (1) follows. For (2), remark that $y=(1/2)\log(a/b)>0$ is acceptable. □

Corollary 4. For $a>b>0$,

$$\frac{G}{A}=\cos(x),\ \frac{H}{A}=\cos(x)^2,\ \frac{P}{A}=\frac{\sin(x)}{x},\ \frac{X}{A}=e^{x\cot(x)-1}, \tag{5}$$

$$\frac{L}{G}=\frac{\sinh(y)}{y},\ \frac{L}{A}=\frac{\tanh(y)}{y},\ \frac{H}{G}=\frac{1}{\cosh(y)},\ \frac{Y}{G}=e^{\tanh(y)/y-1}. \tag{6}$$

where $G=G(a,b)$, $L=L(a,b)$ and $P=P(a,b)$.

Proof. Utilizing Lemma 3 we get, $a.b=\cos(x)^2A^2$, so $\cos(x)=G/A$. Similarly,

$$\frac{P}{A}=\frac{(a-b)/(a+b)}{\arcsin((a-b)/(a+b))}=\frac{\sin(x)}{x}$$

and

$$x\,\text{cot}x-1=(A/P)(G/A)-1=G/P-1,$$

so the last identity follows as well. Also, $L(a,b)=G(e^y+e^{-y})/2=G\sinh(y)$. The other identities in (6) follow in the same manner. □

3 Proofs of main results

This section contains the proofs of our theorems.

Theorem 7. For $a>b>0$, we have

(1) $\frac{1}{e}(G+H)<Y<\frac{1}{2}(G+H)$,

(2) $c \cdot GH < Y^2, \quad c \approx 0.95182.$

Proof. The second inequality in (1) is equivalent to $Y/G < (1 + H/G)/2$, which can be written as

$$e^{\tanh(x)/x} - 1 < \frac{1}{2}\left(1 + 1\frac{1}{\cosh(x)}\right)$$

by (6), and this holds true by Lemma 2(2). Apply the inequality $e^t > 1 + t(t > 0)$ and $\sinh(x) > x$, we get

$$e^{\tanh(x)/x)} > 1 + \frac{\tanh(x)}{x} > 1 + \frac{1}{\cosh(x)},$$

which is equivalent to the first inequality in (1) by (6).

For the proof of (2) remark that this inequality may be rewritten as $(Y/G)^2 > c.(H/G)$, or, by (6) as

$$e^{2(\tanh(x)/x-1)} > \frac{c}{\cosh(x)}, \quad x > 0.$$

The above inequality may be written also as

$$f_2(x) = \log(\cosh(x)) + 2\frac{\tanh(x)}{x} > 2 + \log c = c'. \tag{8}$$

One has

$$x^2 \cosh(x)^2 f_2'(x) = x^2 \cosh(x)\sinh(x) + 2(x - \sinh(x)\cosh(x)) = g(x).$$

By letting $x = t/2\,(t > 0)$,

$$g(x) = \frac{1}{8}(t^2 - 8)\sinh(t) + t = h(t).$$

Clearly, for $t > 2\sqrt{2} \equiv 2.82843$ one has $h(t) > 0$. For $t \in (0, 2\sqrt{2})$, by examining the graphs of elementary functions $a(t) = \sinh(t)/t$ and $b(t) = 8/(8 - t^2)$, we get that there is a unique $t_0 \approx 1.66575$ in $(0, \sqrt{2})$ such that $a(t) = b(t)$ and $a(t) < b(t)$ for $t < t_0$ and $a(t) > b(t)$ for $t > t_0$. This means that $h(t) < 0$ for $t < t_0$ and $h(t) > 0$ for $t > t_0$. In other words, the point $x_0 = t_0/2$ is a minimum point of the function $f_2(x)$, i.e., $f_2(x) \geq f_2(x_0) = c' \approx 1.95062$. Finally, one has $c = e^{c'-2} \approx 0.95182$. □

Remark 9. The inequality $(G + h)/e < Y$ implies

$$\frac{1}{3}(2H + G) < Y. \tag{10}$$

Indeed, this is equivalent to $(3 - e)G > (2e - 3)H$ by $G > H$. Now, the inequality (10) in turn implies $GH^2 < Y^3$, as the arithmetic mean of H, H, and G is greater than their geometric mean.

Our following result improves the classical inequality $G < L$.

Theorem 11.

$$G^2 < \sqrt{\frac{H^2 I^3}{G}} < IY < IG < L^2. \tag{12}$$

Proof. In [11], it is proved that $Y > LG/A$. Since $H = G^2/A$ and $(L/G)^2 > I/G$, we get

$$Y^2 > \frac{H^2 I}{G} = \frac{G^3 I}{A^2}.$$

By inequality $I^3 > A^2 G$ (see [12]), we obtain easily the first two inequalities of (12). For the proof of the last two inequalities of (12), see [11] and [2], respectively. $\square$

Remark 13. By (6), the Identric mean $I(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$ can be deduced to $I/G = e^{x/\tanh(x)-1}$. Combining this with $Y/G = e^{\tanh(x)/x-1}$ we get

$$\log\left(\frac{I}{G}\right) = \frac{A}{L} - 1, \quad \text{and} \quad \log\left(\frac{Y}{G}\right) = \frac{L}{A} - 1. \tag{14}$$

Since $u + 1/u > 2$ for u distinct from 1, we get $\log(I/G) + \log(Y/G) > 0$ from (14). This implies the following inequality:

$$G^2 < IY.$$

Utilizing the inequalities $Y > H$ and $I > \sqrt{AL}$ (see [11–13]) and (12), we get the following relation

$$H\sqrt{AL} < L^2. \tag{15}$$

Using the identity $H = G^2/A$, (15) may be rewritten as

$$G^2 A < L^3,$$

which is a famous inequality of Leach and Sholander. Clearly (12) refines this relation.

Another proof of Theorem 7(1). By identity (14) we get the following identity connecting the means I and Y:

$$1 + \log(Y/G) = 1/(1 + \log(I/G)). \tag{16}$$

In [13], Sándor proved that

$$I > (2A + G)/3,$$

equivalently

$$1 + \log\left(\frac{I}{G}\right) > 1 + \log\left(\frac{1+2x}{3}\right),$$

where $x = A/G > 1$. Now, as $H = G^2/A$, inequality (1) may be written as

$$\log\left(\frac{Y}{G}\right) < \log\left(\frac{1+1/x}{2}\right) = \log\left(\frac{2x}{1+x}\right).$$

By (16) one has

$$\log\left(\frac{Y}{G}\right) < 1/\left(1 + \log\left(\frac{1+2x}{3}\right)\right) - 1 < \log\left(\frac{2x}{1+x}\right),$$

equivalently

$$\left(1 + \log\left(\frac{2x}{1+x}\right)\right) \cdot \left(1 + \log\left(\frac{1+2x}{3}\right)\right) > 1,$$

which holds true by Lemma 1 (2).

Corollary 17. For $a > b > 0$, one has the following double inequality

$$\frac{(a-b)^2}{4(a^2+ab+b^2)} < \log\left(\frac{G}{Y}\right) < \frac{(a-b)^2}{a^2+10ab+b^2}.$$

Proof. It is proved in [14]

$$\frac{1}{3}\frac{(a-b)^2}{(a+b)^2} < \log\left(\frac{I}{G}\right) < \frac{1}{12}\frac{(a-b)^2}{ab}.$$

By using identity (11), after certain elementary transformations, we get the desired inequalities. □

Corollary 18. For $a > b > 0$, one has

1. $\log\left(\frac{X}{A}\right) < \frac{\pi}{2}\frac{G}{A} - 1,$

2. $2\left(1-\frac{A}{P}\right) < \log\left(\frac{X}{A}\right) < \left(\frac{P}{A}\right)^2.$

Proof. By definition, $\log(X/A) = G/P - 1$. Now, it is well-known that $A/P < pi/2$ (see [15]). Observing $G/P = (G/A)(A/P)$, since (1) follows. Double inequality in (2) follows from the identity for $\log(X/A)$ and the double inequality $A^2G < P < (2A+G)/3$, (see [15]). □

Proof of Theorem B. For the following series representation

$$\frac{A}{P} = \sum_{n=0}^{\infty}\frac{n(2n,n)}{4^n(2n+1)}t^n, \quad \frac{A}{G} = \sum_{n=0}^{\infty}\frac{(2n,n)}{4^n}t^n, \tag{19}$$

see [18]. Applying the following identities

$$\frac{A}{P} = \frac{A}{G}\left(2 - \log\left(\frac{X}{A}\right)\right),$$

$$\log\left(\frac{X}{A}\right) = \frac{G}{A}\left(\frac{A}{P} - \frac{A}{G}\right),$$

together with (19), we get (1).

For (2), we use the series representation (see Sándor, 1993, [14]):

$$1 + \log\left(\frac{I}{G}\right) = \sum_{n=0}^{\infty} c_n t^n,$$

where $c_n = 1/(2n+1)$. Applying the identity $1 + log(Y/G) = 1/(1 + log(I/G))$ we will write

$$1/\sum_{n=0}^{\infty} c_n t^n$$

as a power series

$$1 + b_1 t + b_2 t^2 + b_3 t^3 + \ldots + b_n t^n + \ldots.$$

This implies that the product of power series with coefficients c_n as well as b_n is equal to 1. By making the product of two power series, and identifying the coefficients (i.e. all new coefficient with indexes $n \geq 1$ will be zero), we get:

$$\frac{1}{3} + a_1 = 0,\ \frac{1}{5} + a_{1/3} + a_5 = 0, \ldots, 1/(2n+1) + a_{1/(2n-1)} + a_{2/(2n-3)} + \ldots + a_{(n-1)/3} + a_n = 0.$$

These relations will give recurrently all coefficients b_n. □

Theorem 20. One has

(1) $\dfrac{G-Y}{A-L)} < \dfrac{Y+G}{2A} < \dfrac{3G+H}{4A} < 1,$

(2) $L < \dfrac{2G+A}{3} < X < L(X,A) < P < \dfrac{2A+G}{3} < I.$

Proof. Clearly, $L(Y,G) = (G-Y)A/(A-L)$. By the definition of Y, and utilizing the inequalities $L < (A+G)/2$ and $Y < (G+H)/2$, we the first and second inequality in (1). The last inequality of (1) is obvious, since G and $H < A$.

For (2), the first inequality is due to B. C. Carlson (see [4, 12]), while the second one appears in [11]. The last two inequalities are proved in [15] and [14], respectively. The third inequality is obvious because $X < A$. So we have to prove only $L(X,A) < P$. By the inequality $A - X < P - G$ [11, Theorem 2.10], we get

$$L(X,A) = \frac{A-X}{\log(A/X)} = \frac{(A-X)P}{P-G} < P.$$

□

From the part (1) of the above theorem we get $Y > L + G - A$, which is similar to $X > A + G - P$.

Remark 21. Neuman–Sándor [9] proved that $L(G,A) > L$. Since $X > G$, we easily get $L(X,A) > L(G,A) > L$, so we get the following refinement of inequality $L < P$:

$$L < L(G,A) < L(X,A) < P.$$

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