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On the tree of the General Euclidean Algorithm

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Abstract: The General Euclidean Algorithm (GEA) is the natural generalization of the Euclidean Algorithm (EA) and equivalent to Semi Regular Continued Fractions (SRCF). In this paper, we consider the finite case with entries in GEA natural numbers. Consider the Euclidean Division. In GEA we want the divider to be bigger than the absolute value of the remainder. Thus, we take two divisions except for the case the remainder is zero. However, for our help, we consider the non Euclidean remainder without the negative sign so as not to take the absolute value of it for the next step of the algorithm as we need to have two positive integers. So, it occurs a binary tree except for the before last vertex of its path which gives one division as the remainder is zero. This paper presents mainly a criterion with which we can find all the shortest paths of this tree and not only the one that Valhen–Kronecker's criterion [3] gives. In terms of SRCF, this criterion gives all the SRCF expansions of a rational number t with the same length as the Nearest Integer Continued Fraction (NICF) expansion of t. This criterion, as we shall see, is related to the golden ration. Afterwards, it is presented a theorem which connects the Fibonacci sequence with the difference between the longest and the shortest path of this tree, a theorem which connects the Fibonacci sequence with the longest path of this tree and a different proof of a theorem which occurs by [1] and [3] which connects the pell numbers with the shortest path of the aforementioned tree. After that, it is proven a connection of this tree to the harmonic and the geometric mean and in particular two new criteria of finding a shortest path are constructed based on this two means. In the final chapter, it is an algorithm, which has an "opposite" property of the EA, property which has been proven in [2] and has to do with the number of steps Least Remainder Algorithm (LRA) needs to be finished in relation to EA and the signs of the remainders of LRA path.
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1 Introduction

We begin by defining the GEA, give an example and present the notation we will use for the rest of this paper. It is easy to prove the equivalence of GEA and SRCF.

Let $a \in \mathbb{N}, b \in \mathbb{N}, a > b$. Also, for the algorithm to proceed beyond the first step, we suppose that $b \nmid a$. Then there is $q \in \mathbb{N}$ such that

$$qb < a < (q+1)b \tag{1.1}$$

We now set

$$r = a - qb \tag{1.2}$$

for which we show easily that

$$0 < \{r, b-r\} < b \tag{1.3}$$

Now we can write down the first step of the General Euclidean Algorithm (GEA) for the pair (a,b), which is

$$\begin{gathered} a = qb + r,\ 0 < r < b \\ a = (q+1)b - (b-r),\ 0 < b-r < b \end{gathered} \tag{1.4}$$

What we did for (a, b), we do now for the pairs $(b, r), (b, b-r)$ and we continue until the remainders are zero.

Lets see an example by applying the GEA for the pair (8, 5)

$$\begin{tikzcd}
 & & 3 = 1 \cdot 2 + 1 \arrow[r] & 2 = 2 \cdot 1 + 0 \\
 & 5 = 1 \cdot 3 + 2 \arrow[ur] \arrow[dr] & & \\
8 = 1 \cdot 5 + 3 \arrow[ur] \arrow[dr] & & 3 = 2 \cdot 2 - 1 \arrow[r] & 2 = 2 \cdot 1 + 0 \\
 & 5 = 2 \cdot 3 - 1 \arrow[r] & 3 = 3 \cdot 1 + 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 5 = 2 \cdot 2 + 1 \arrow[r] & 2 = 2 \cdot 1 + 0 \\
8 = 2 \cdot 5 - 2 \arrow[ur] \arrow[dr] & & \\
 & 5 = 3 \cdot 2 - 1 \arrow[r] & 2 = 2 \cdot 1 + 0
\end{tikzcd}$$

Now we substract the leaves and we add a root vertex, let $13 = 1 \cdot 8 + 5$. Thus, the Euclidean tree $< 8, 5 >$ assumes the form

$$\begin{tikzcd}
 & & & 3 = 1 \cdot 2 + 1 \\
 & & 5 = 1 \cdot 3 + 2 \arrow[ur] \arrow[dr] & \\
 & 8 = 1 \cdot 5 + 3 \arrow[ur] \arrow[dr] & & 3 = 2 \cdot 2 - 1 \\
13 = 1 \cdot 8 + 5 \arrow[ur] \arrow[ddr] & & 5 = 2 \cdot 3 - 1 & \\
 & & 5 = 2 \cdot 2 + 1 & \\
 & 8 = 2 \cdot 5 - 2 \arrow[ur] \arrow[dr] & & \\
 & & 5 = 3 \cdot 2 - 1 &
\end{tikzcd}$$

It is clear that the number of the directed paths and the length of each, which is what we will examine in this paper, do not change.

Knowing the divisor and the remainder of a vertex, we can find its children and hence we can write the $< 8, 5 >$ as

$$\begin{tikzcd}
& & & {[2,1]} \\
& & {[3,2]} \arrow[ur] \arrow[dr] & \\
& {[5,3]} \arrow[ur] \arrow[dr] & & {[2,1]} \\
{[8,5]} \arrow[ur] \arrow[ddr] & & {[3,1]} & \\
& & {[2,1]} & \\
& {[5,2]} \arrow[ur] \arrow[dr] & & \\
& & {[2,1]} &
\end{tikzcd}$$

Euclidean tree $< 8, 5 >$

This is how we will write the Euclidean Trees from now on. Also, for the remainder of this paper, we shall mean "directed path" whenever we say "path" and "Euclidean tree" whenever we say "tree".

Hence (1.4) becomes

$$\begin{tikzcd}
& {[b,r]} \\
{[a,b]} \arrow[ur] \arrow[dr] & \\
& {[b,b-r]}
\end{tikzcd}$$

$Tree\ 1$

$(Tree\ < a, b >\ first\ step)$

2 Basic definitions and lemmas

In the second chapter we give some definitions and prove some lemmas which we will use extensively throughout this paper.

Lemma 2.1. *Consider the tree* $< a_1, a_2 >$ *and a path of it, the*

$$[a_1, a_2] \to \ldots \to [a_{\rho-1}, a_\rho] \to [a_\rho, a_{\rho+1}]$$

where $\rho \geq 4$ *and* $[a_\rho, a_{\rho+1}]$ *leaf of* $< a_1, a_2 >$.

***1**. If there is* $\lambda \in \mathbb{N}$ *with* $1 < \lambda \leq \rho - 2$ *such that* $[a_\lambda, a_{\lambda+1}] \to \ldots \to [a_\rho, a_{\rho+1}]$ *is not a shortest path (SP) of subtree* $< a_\lambda, a_{\lambda+1} >$ *(respectively longest path), then for every* $k \in \mathbb{N}$ *with* $1 \leq k < \lambda$ *we have that the paths* $[a_k, a_{k+1}] \to \ldots \to [a_\rho, a_{\rho+1}]$ *aren't SP (respectively LP) of* $< a_k, a_{k+1} >$.

2. If the path of the assertion is a SP of $< a_1, a_2 >$ (respectively LP), then for every $\lambda \in \mathbb{N}$ with $1 < \lambda \leq \rho - 2$, the paths $[a_\lambda, a_{\lambda+1}] \to \ldots \to [a_\rho, a_{\rho+1}]$ are SP of $< a_\lambda, a_{\lambda+1} >$ (respectively LP).

Proof. For the first: Let $\lambda \in \mathbb{N}$ with $1 < \lambda \leq \rho - 2$ such that $[a_\lambda, a_{\lambda+1}] \to \ldots \to [a_\rho, a_{\rho+1}]$ is not a SP of $< a_\lambda, a_{\lambda+1} >$. Then we choose one which is, the

$$[a_\lambda, a_{\lambda+1}] \to [a_{\lambda+1}, a'_{\lambda+2}] \to \ldots \to [a'_m, a'_{m+1}]$$

Then for every $k \in \mathbb{N}$ with $1 \leq k < \lambda$, the path

$$[a_k, a_{k+1}] \to \ldots \to [a_\lambda, a_{\lambda+1}] \to \ldots \to [a'_m, a'_{m+1}]$$

is shorter than

$$[a_k, a_{k+1}] \to \ldots \to [a_\lambda, a_{\lambda+1}] \to \ldots \to [a_\rho, a_{\rho+1}]$$

and hence the latter is not SP of $< a_k, a_{k+1} >$. We do the same in case it is LP.

For the second: We have that $[a_1, a_2] \to \ldots \to [a_\rho, a_{\rho+1}]$ is LP of $< a_1, a_2 >$. Suppose now there is $\lambda_0 \in \mathbb{N}$ with $1 < \lambda_0 \leq \rho - 2$ such that $[a_{\lambda_0}, a_{\lambda_0+1}] \to \ldots \to [a_\rho, a_{\rho+1}]$ is not LP of $< a_{\lambda_0}, a_{\lambda_0+1} >$. Then we choose one which is, the

$$[a_{\lambda_0}, a_{\lambda_0+1}] \to [a_{\lambda_0+1}, a'_{\lambda_0+2}] \to \ldots \to [a'_m, a'_{m+1}]$$

Then the path

$$[a_1, a_2] \to \ldots \to [a_{\lambda_0}, a_{\lambda_0+1}] \to \ldots \to [a'_m, a'_{m+1}]$$

is longer than the given one which results in a contradiction. We do the same in case it is SP. $\square$

Before we continue to the central theorem of this paper, we will need two definitions and lemmas.

Definition 2.2. *Let $a \in \mathbb{N}, b \in \mathbb{N}, q \in \mathbb{N}$ with (1.1),(1.2). Now let $k \in \mathbb{Z}$ with $k < q$ and $a' = a - kq$. The first step of GEA of (a', b) is*

$$a' = (q - k)b + r, \; 0 < r < b$$
$$a' = (q - k + 1)b - (b - r), \; 0 < b - r < b$$

or

$$\begin{tikzcd} & {[b, r]} \\ {[a', b]} \arrow[ur] \arrow[dr] & \\ & {[b, b - r]} \end{tikzcd}$$

We call $< a', b >$, $< a, b >$ equivalent trees.

We notice immediately that

Lemma 2.3. *Let the equivalent trees* $< a, b >$, $< a', b >$. *Then the children of the root vertex are the same and also for every vertex of* $< a, b >$ *except for the root vertex, we have that: If it is found in SP of* $< a, b >$ *(respectively in LP) then it is found in SP of* $< a', b >$ *(respectively in LP) and the converse.*

Definition 2.4. *Let* $a \in \mathbb{N}, b \in \mathbb{N}, q \in \mathbb{N}$ *with (1.1),(1.2). Then* $qb < a < (q+1)b$. *Equivalently* $qb < (2q+1)b - a < (q+1)b$ *and hence the first step of GEA of* $((2q+1)b - a, b)$ *is*

$$(2q+1)b - a = qb + (b - r),\ 0 < b - r < b$$
$$(2q+1)b - a = (q+1)b - r,\ 0 < r < b$$

or

$$\begin{array}{ccc} & & [b, b-r] \\ & \nearrow & \\ [(2q+1)b - a, b] & & \\ & \searrow & \\ & & [b, r] \end{array}$$

We call $< a, b >$, $< (2q+1)b - a, b >$ *inverse trees.*

We notice immediately that

Lemma 2.5. *Let the inverse trees* $< a, b >$, $< (2q+1)b - a, b >$. *Then the children of two root vertices are the same and also for every vertex of* $< a, b >$ *except for the root vertex, we have that: If it is not found in SP of* $< a, b >$ *(respectively in LP), then it is not found in SP of* $< (2q+1)b - a, b >$ *(respectively in LP) and the converse.*

3 The main Theorem with some Corollaries

This chapter is dedicated to the main theorem of this paper and some corollaries. This theorem gives us all the shortest paths of the Euclidean Tree instead of the one Valhen–Kronecker's theorem gives us. Also the state of the theorem by itself provides us a way to treat other problems as we shall see in the rest of the paper. By putting $q = 1$, golden ratio appears.

Theorem 3.1. *(<u>**Master Theorem**</u>) Let* $a \in \mathbb{N}, b \in \mathbb{N}, q \in \mathbb{N}$, $qb < a < (q+1)b$, $r = a - qb$. *Then*
I. *If* $(q + \frac{-1+\sqrt{5}}{2})b < a < (q+1)b$, *then the vertex* $[b, r]$ *is not found in SP of* $< a, b >$ *and* $[b, b-r]$ *is not found in LP of* $< a, b >$.
II. *If* $qb < a < (q + \frac{3-\sqrt{5}}{2})b$, *then* $[b, b-r]$ *is not found in SP of* $< a, b >$ *and* $[b, r]$ *is not found in LP of* $< a, b >$.
IIIa. *If* $(q + \frac{1}{2})b < a < (q + \frac{-1+\sqrt{5}}{2})b$, *then* $[b, b-r]$ *is not found in LP of* $< a, b >$ *and* $[b, r]$, $[b, b-r]$ *are found in SP of* $< a, b >$.
IIIb. *If* $(q + \frac{3-\sqrt{5}}{2})b < a < (q + \frac{1}{2})b$, *then* $[b, r]$ *is not found in LP of* $< a, b >$ *and* $[b, r]$, $[b, b-r]$ *are found in SP of* $< a, b >$.

IV*. If* $(q+\frac{1}{2})b = a$*, then*

$$
\begin{tikzcd}
& {[b,1]} \\
{[a,b]} \arrow[ur] \arrow[dr] & \\
& {[b,1]}
\end{tikzcd}
$$

Proof. We will reduce the proof by proving the following assertion.

Assertion. It is enough to show the cases IV, I, IIIa.

Proof of the Assertion. The case IV is trivial. Now we suppose that the theorem holds for the case I and let the pair (a, b) be in case II, that is, $qb < a < \left(q + \frac{3-\sqrt{5}}{2}\right) b$, $q \in \mathbb{N}$ and equivalently $\left(q + \frac{-1+\sqrt{5}}{2}\right) b < (2q+1)b - a < (q+1)b$ which means that the pair (a', b') where $a' = (2q+1)b - a$, $b' = b$, $q' = q$, $r' = a' - q'b' = b - r$ and $b' - r' = r$, is in case I. By applying the theorem for the pair (a', b') we have that $[b', r'] = [b, b-r]$ is not in SP of $< a', b' >$ and $[b', b'-r'] = [b, r]$ is not in LP of $< a', b' >$ and by Lemma 2.5, we have that $[b, r]$ is not in LP of $< a, b >$ and $[b, b-r]$ is not in SP of $< a, b >$. Thus if the theorem holds for the case I, then it holds for the case II.

We do the same for the case IIIb by supposing that the theorem holds for the case IIIa. $\square$

Now we continue with the proof of Master Theorem. We will apply induction on b. The smallest b for which there is $a \in \mathbb{N}$: (a, b) is in case I, is $b = 3$. Then $a = 2k + 3$, $k \in \mathbb{N}$ and by Lemma 2.3 we can take $k = 1$:

$$
\begin{tikzcd}
& & {[2,1]} \\
& {[3,2]} \arrow[ur] \arrow[dr] & \\
{[5,3]} \arrow[ur] \arrow[dr] & & {[2,1]} \\
& {[3,1]} &
\end{tikzcd}
$$

where we can easily confirm the assertion of the theorem.

Now the smallest b for which there is $a \in \mathbb{N}$: (a, b) is in case IIIa, is $b = 5$. Then $a = 3k + 5,\ k \in \mathbb{N}$ and by Lemma 2.3 we can take $k = 1$:

$$
\begin{tikzcd}
& & & {[2,1]} \\
& & {[3,2]} \arrow[ur] \arrow[dr] & \\
& {[5,3]} \arrow[ur] \arrow[dr] & & {[2,1]} \\
{[8,5]} \arrow[ur] \arrow[ddr] & & {[3,1]} & \\
& & {[2,1]} & \\
& {[5,2]} \arrow[ur] \arrow[dr] & & \\
& & {[2,1]} &
\end{tikzcd}
$$

where we can easily confirm the assertion of the theorem.

Now, due to the Assertion above, we don't have to examine the cases II and, IIIb. Also it is enough to confirm the theorem for these two pairs $(5,3)$ and $(8,5)$ before we apply the induction but this can be seen only through the rest of the theorem. Now let us assume that the theorem is true for all cases when the divisor is less than b.

<u>Case I</u> In this case we have $(q + \frac{-1+\sqrt{5}}{2})b < a < (q+1)b$. By replacing $a = qb + r$ we take

$$r < b < \frac{1+\sqrt{5}}{2} r \tag{3.5}$$

The relationship (3.5) implies that the pair (b, r) either is in case II or IIIa or IIIb. We write

$$(b, r) \in \{II, III\} \tag{3.6}$$

In addition, (3.5) implies $r < b < 2r$ from which we take the first step of GEA for (b, r):

$$b = 1 \cdot r + (b - r),\ 0 < b - r < r$$
$$b = 2 \cdot r - (2r - b),\ 0 < 2r - b < r$$

or

$$\begin{tikzcd}
& {[r, b-r]} \\
{[b, r]} \arrow[ur] \arrow[dr] & \\
& {[r, 2b-r]}
\end{tikzcd}$$

We combine the last tree with Tree 1

$$\begin{tikzcd}
& & {[r, b-r]} \\
& {[b, r]} \arrow[ur] \arrow[dr] & \\
{[a, b]} \arrow[ur] \arrow[dr] & & {[r, 2r-b]} \\
& {[b, b-r]} &
\end{tikzcd}$$

Tree 2

As $b = r \mod (b-r)$, the subtrees of the Tree 2, $< b, b-r >$, $< r, b-r >$ are equivalent and hence if the $[b, b-r]$ was in LP of $< a, b >$, then by Lemma 2.3, the same should be happen for $[r, b-r]$ which is a contradiction as $[r, b-r]$ is in the second step of the algorithm and $[b, b-r]$ is in the first.

Now we will show that $[b, r]$ is not found in SP of $< a, b >$. To show that we have to prove that the paths

$$[a, b] \to [b, r] \to [r, b-r] \to \dots$$
$$[a, b] \to [b, r] \to [r, 2r-b] \to \dots$$

aren't SP of $< a, b >$. The first path cannot be a SP of $< a, b >$ as $< r, b-r >$, $< b, b-r >$ are equivalent trees and in a different step of Tree 2 (see previous paragraph).

So it remains to show that the path

$$[a, b] \to [b, r] \to [r, 2r - b] \to \dots$$

is not a SP of $< a, b >$. Now from (3.6) we suppose that $(b, r) \in$ II and afterwards we will suppose that $(b, r) \in$ III. So from the hypothesis that for every number less than b the theorem is true, we take that $[r, 2r - b]$ is not in SP of $< b, r >$ and immediately by Lemma 2.1, $[r, 2r - b]$ is not in SP of $< a, b >$ and hence this path is not a SP of $< a, b >$. Now we suppose that $(b, r) \in$ III. Then by inductive hypothesis $[r, b - r], [r, 2r - b]$ are in SP of $< b, r >$. If now we suppose that $[r, 2r - b]$ is in SP of $< a, b >$, then the same thing must be happen for its sibling $[r, b - r]$ as they are at the same step of the algorithm and they both are in SP of $< b, r >$, but, in the previous paragraph, we showed that $[r, b - r]$ is not SP of $< a, b >$.

<u>Case IIIa</u> In this case we have $\left(q + \frac{1}{2}\right) b < a < \left(q + \frac{-1+\sqrt{5}}{2}\right) b$. By replacing $a = qb + r$ we take

$$\left(1 + \frac{-1 + \sqrt{5}}{2}\right) r < b < 2r \tag{3.7}$$

Equivalently $2(b - r) < b < \left(2 + \frac{-1+\sqrt{5}}{2}\right)(b - r)$ and that implies that the pair $(b, b - r)$ is either in case II or III

$$(b, b - r) \in \{II, III\} \tag{3.8}$$

Also from (3.7) we take

$$r < b < 2r \tag{3.9}$$

$$2(b - r) < b < 3(b - r) \tag{3.10}$$

From (3.9) we take Tree 2 (see the proof for Case I) and from (3.10) we take the first step of the GEA for $(b, b - r)$:

$$b = 2(b - r) + (2r - b),\ 0 < 2r - b < b - r$$

$$b = 3(b - r) - (2b - 3r),\ 0 < 2b - 3r < b - r$$

or

$$\begin{tikzcd} & {[b - r, 2r - b]} \\ {[b, b - r]} \arrow[ur] \arrow[dr] & \\ & {[b - r, 2b - 3r]} \end{tikzcd}$$

Now we combine the last tree with Tree 2

$$\begin{tikzcd} & & {[r, b - r]} \\ & {[b, r]} \arrow[ur] \arrow[dr] & \\ {[a, b]} \arrow[ur] \arrow[ddr] & & {[r, 2r - b]} \\ & & {[b - r, 2r - b]} \\ & {[b, b - r]} \arrow[ur] \arrow[dr] & \\ & & {[b - r, 2b - 3r]} \end{tikzcd}$$

Tree 3

In this case IIIa we have to prove that $[b, b-r]$ is not found in LP of $< a, b >$ and $[b, r]$, $[b, b-r]$ are found in SP of $< a, b >$. The proof of $[b, b-r]$ not being in LP of $< a, b >$ is the same as in Case I.

Now, in order to prove that $[b, r]$, $[b, b-r]$ are in SP of $< a, b >$ we have to show that at least one of the two paths

$$[a, b] \to [b, r] \to [r, b-r] \to \dots$$
$$[a, b] \to [b, r] \to [r, 2r-b] \to \dots$$

is a SP of $< a, b >$ and at least one of the two paths

$$[a, b] \to [b, b-r] \to [b-r, 2r-b] \to \dots$$
$$[a, b] \to [b, b-r] \to [b-r, 2b-3r] \to \dots$$

is a SP of $< a, b >$. $[r, b-r]$ is not in SP of $< a, b >$ (see proof of Case I) and hence we immediately exclude $[a, b] \to [b, r] \to [r, b-r] \to \dots$ from possible paths that are SP of $< a, b >$.

As $r \equiv b - r \mod (2r - b)$, the subtrees $< r, 2r-b >$, $< b-r, 2r-b >$ of Tree 3 are equivalent. Thus from Lemma 2.3 and because there are in the same step of the algorithm, we have that either both $[r, 2r-b]$, $[b-r, 2r-b]$ is in SP of $< a, b >$ or none of them. If both are, then $[a, b] \to [b, r] \to [r, 2r-b] \to \dots$ and $[a, b] \to [b, b-r] \to [b-r, 2r-b] \to \dots$ are SP of $< a, b >$ and hence $[b, r]$, $[b, b-r]$ are in SP of $< a, b >$ and so Case IIIa is proven. We now consider that both $[r, 2r-b]$, $[b-r, 2r-b]$ are in SP of $< a, b >$ and we will make a contradiction.

So, let $[r, 2r-b]$, $[b-r, 2r-b]$ not be in SP of $< a, b >$. Thus, it remains the path $[a, b] \to [b, b-r] \to [b-r, 2b-3r] \to \dots$ as the unique path from the four paths above which must be SP of $< a, b >$. Then by Lemma 2.1, $[b, b-r] \to [b-r, 2b-3r] \to \dots$ is SP of $< b, b-r >$ and so $[b-r, 2b-3r]$ is in SP of $< b, b-r >$.

Now we will use the relation (3.8) to make a contradiction. From (3.8) we suppose $(b, b-r) \in$ II and hence from the inductive hypothesis $[b-r, 2b-3r]$ is not in SP of $< b, b-r >$. And finally we suppose from (3.8), $(b, b-r) \in$ III. Then by inductive hypothesis, both $[b-r, 2r-b]$, $[b-r, 2b-3r]$ are in SP of $< b, b-r >$ and because $[b-r, 2b-3r]$ is in SP of $< a, b >$ we take that the same thing must be happen to its sibling $[b-r, 2r-b]$ which contradicts to our assumption.

Corrollary 3.2. *Let the tree $< a, b >$ with $ab < a < (a+1)b$, $q \in \mathbb{N}$, $r = a - qb$ and $\max(a, b)$ is the length of a LP of $< a, b >$ and $\min(a, b)$ is the length of a SP of $< a, b >$. Then*
1. The next three are equivalent:
1.1 $(a, b) \in I$
1.2 $\min(a, b) = \min(b, r) = \min(b, b-r) + 1,$
$\max(a, b) = \max(b, r) + 1$
1.3 $\min(a, b) = \min(b, r)$

2. The next three are equivalent:
2.1 $(a, b) \in IIIa$
2.2 $\min(a, b) = \min(b, r) + 1 = \min(b, b-r) + 1,$

$\max(a,b) = \max(b,r) + 1$
2.3 $\min(b,r) = \min(b, b-r)$,
$\max(a,b) = \max(b,r) + 1$

3. The next three are equivalent:
3.1 $(a,b) \in IIIb$
3.2 $\min(a,b) = \min(b,r) + 1 = \min(b, b-r) + 1$,
$\max(a,b) = \max(b, b-r) + 1$
3.3 $\min(b,r) = \min(b, b-r)$,
$\max(a,b) = \max(b, b-r) + 1$

4. The next three are equivalent:
4.1 $(a,b) \in II$
4.2 $\min(a,b) = \min(b, b-r) + 1 = \min(b,r) + 1$,
$\max(a,b) = \max(b, b-r) + 1$
4.3 $\min(a,b) = \min(b, b-r)$

Proof. We only have to prove the first and the second and to show that, we use the same arguments as for the *Assertion* in the beginning of the proof of the Master Theorem.

For 1. (1.1⇒1.2). We have that $(a,b) \in$ I and hence by the Master Theorem, $[b,r]$ is not in SP of $< a,b >$ and $[b, b-r]$ is not in LP of $< a,b >$ and so, as at least one of the siblings must be necessarily SP or LP of $< a,b >$, $[b, b-r]$ is in SP of $< a,b >$ and $[b,r]$ in LP. Then, as $[b,r]$, $[b, b-r]$ are the children of $[a,b]$, it is clear that $\min(a,b) = \min(b, b-r) + 1$, $\max(a,b) = \max(b,r) + 1$.

In addition, in Case I, we have Tree 2 and the relationship (3.6) which is $(b,r) \in \{$II, III$\}$. Then from the Master Theorem $[r, b-r]$ is in SP of $< b,r >$ and hence $\min(b,r) = \min(r, b-r) + 1$. As now $< r, b-r >$, $< b, b-r >$ are equivalent trees, by Lemma 2.3 we take $\min(r, b-r) = \min(b, b-r)$ and hence $\min(b,r) = \min(b, b-r) + 1$.

(1.2⇒1.3). Trivial

(1.3⇒1.1). Let $(a,b) \in \{$II, III$\}$. Then by the Master Theorem $[b,r]$ is in SP of $< a,b >$ and hence $\min(a,b) = \min(b,r) + 1$ which contradicts our assumption.

For 2. The proof for 2 is the same as for 1, except for (2.3⇒2.1). So let $\min(b,r) = \min(b, b-r)$, $\max(a,b) = \max(b,r) + 1$. Now we assume that $(a,b) \notin$ IIIa and hence $(a,b) \in \{$II, IIIb, II$\}$.

$Let (a,b) \in$ I. Then by (1.1⇒1.2) we take $\min(b,r) = \min(b, b-r) + 1$ which contradicts the hypothesis.

Let now $(a,b) \in \{$ II, IIIb $\}$. Then by the Master Theorem, $[b,r]$ is not in LP of $< a,b >$ and $[b, b-r]$ is in LP of $< a,b >$ and hence $\max(b,r) < \max(b, b-r)$ and $\max(a,b) = \max(b, b-r) + 1$ and so $\max(b,r) < \max(b, b-r) = \max(a,b) - 1 = \max(b,r)$ and we arrive at a contradiction.

Corollary 3.3. *Let* $< a, b >$ *with* $qb < a < (q+1)b,\ q \in \mathbb{N}, r = a - qb$ *and* $A(a,b) = \max(a,b) - \min(a,b)$. *Then*
1. If $(a,b) \in I$*, then* $A(a,b) = A(b,r) + 1$.
2. If $(a,b) \in IIIa$*, then* $A(a,b) = A(b,r) = A(b,b-r) + 1$.
3. If $(a,b) \in II$*, then* $A(a,b) = A(b,b-r) + 1$.
4. If $(a,b) \in IIIb$*, then* $A(a,b) = A(b,b-r) = A(b,r) + 1$.

Proof. We only need to prove 1. and 2. (as previously).

For 1. We have from Corollary 3.2 that $\min(a,b) = \min(b,r),\ \max(a,b) = \max(b,r) + 1$ and hence $A(a,b) = \max(a,b) - \min(a,b) = \max(b,r) - \min(b,r) + 1 = A(b,r) + 1$.

For 2. We have from Corollary 3.2 that $\min(a,b) = \min(b,r)+1,\ \max(a,b) = \max(b,r)+1$ and hence $A(a,b) = \max(b,r) - \min(b,r) = A(b,r)$.

Now we will prove that $A(b,r) = A(b,b-r) + 1$. In case IIIa we have also Tree 3 and (3.7) which implies that $(b,r) \in$ I and hence, by part 1. of this Corollary, we take $A(b,r) = A(r,b-r) + 1$. As now $< r, b-r >, < b, b-r >$ are equivalent trees, by Lemma 2.3 we take $\max(r,b-r) = \max(b,b-r),\ \min(r,b-r) = \min(b,b-r)$ and hence $A(r,b-r) = A(b,b-r)$ and hence $A(b,r) = A(b,b-r) + 1$. $\square$

Corollary 3.4. *Every tree has two LP.*

Proof. Let $< a, b >$. Then by the Master Theorem, except for case IV, one of the children of $[a,b]$ is in LP. So we choose the vertex which is in LP of $< a, b >$ and we continue by doing the same with every new vertex. Finally, we reach a vertex which is in case IV, because otherwise the process would not end. Now, we know by the Master Theorem in this case, that there are two paths with one vertex each, and hence we take the two LP for every tree. $\square$

4 Three Theorems on the relation between the GEA, Fibonacci and Pell numbers

This chapter presents three interesting theorems. The first two have to do with the Fibonacci sequence, the Longest and Shortest Paths of the Euclidean Tree. The third is a combination of [1] and [3] and here we prove it by using the tools that Theorem 3.1 gives us and has to do with Pell numbers and the Shortest Paths of the Euclidean Tree.

Lemma 4.1. *Consider* $< a, b >$ *under (1.2),(1.3). We know that* $r, b-r$ *are the two remainders in the first step of the GEA for* (a,b). *Then*
1. If $r' = max\{r, b-r\}$*, then* $\max(a,b) = \max(b,r') + 1$. *If also* $(a,b) \in \{I, II\}$ *then* $\min(a,b) = \min(b,r')$ *and hence* $A(a,b) = A(b,r') + 1$*, and if* $(a,b) \in III$*, then* $\min(a,b) = \min(b,r') + 1$.
2. If $r'' = min\{r, b-r\}$*, then* $\min(a,b) = \min(b,r'') + 1$. *If also* $(a,b) \in III$*, then* $A(a,b) = A(b,r'') + 1$.

Proof. We will only do the proof for the cases I, IIIa. Let $(a, b) \in$ I. Then by (3.5), $r' = r$, $r'' = b - r$, and by Corollary 3.2, we take

$$\begin{aligned}
&\max(a,b) = \max(b,r) + 1 = \max(b,r') + 1\\
&\min(a,b) = \min(b,r) + 1 = \min(b,r') + 1\\
&\min(a,b) = \min(b,b-r) + 1 = \min(b,r'') + 1\\
&A(a,b) = \max(a,b) - \min(a,b) = \max(b,r') - \min(b,r') = A(b,r') + 1
\end{aligned}$$

Let now $(a, b) \in$ IIIa. Then by (3.9), $r' = r, r'' = b - r$ and by Corollary 3.2,

$$\begin{aligned}
&\max(a,b) = \max(b,r) + 1 = \max(b,r') + 1\\
&\min(a,b) = \min(b,r) + 1 = min(b,r') + 1\\
&\min(a,b) = \min(b,b-r) + 1 = \min(b,r'') + 1
\end{aligned}$$

and by Corollary 3.3

$$A(a,b) = A(b,b-r) + 1 = A(b,r'') + 1$$

Lemma 4.2. *Consider the tree $< a, b >$ under (1.1),(1.2): $qb < a < (q + 1)b,\ r = a - qb$. We know that $r, b - r$ are the two remainders in the first step of the GEA for (a, b). Let $r' = \max\{r, b - r\}$, $r'' = \min\{r, b - r\}$. Also let F_n be the Finonacci sequence with $F_1 = F_2 = 1$ and $\beta_{n+2} = 2\beta_{n+1} + \beta_n$ with $\beta_1 = 2,\ \beta_2 = 5$. Then*
1. Let $b > F_{n+2}$. Then if $(a, b) \in III$, then $F_n < r''$ and if $(a, b) \in \{I, II\}$ then $F_{n+1} < r'$.
2. Let $b < \beta_{n+2}$. If $(a, b) \in III$, then $r'' < \beta_{n+2}/2$ and if $(a, b) \in \{I, II\}$, then $r'' < \beta_{n+1}$.

Proof. We will only do the proof for the cases I, IIIa.

For 1. Let $(a, b) \in$ I. Then by (3.7) we have $b < \frac{1+\sqrt{5}}{2}$ and equivalently $r > b\frac{-1+\sqrt{5}}{2}$. Also from the assertion we have $b \geq 1 + F_{n+2}$ and equivalently $b\frac{-1+\sqrt{5}}{2} \geq (1 + F_{n+2})\frac{-1+\sqrt{5}}{2}$, and thus $r > (1 + F_{n+2})\frac{-1+\sqrt{5}}{2}$. In Lemma 4.3 we will show that $1 + F_{n+2} > \frac{1+\sqrt{5}}{2}F_{n+1}$ and so $r > F_{n+1}$. Also from (3.5) we take $r' = r$. Let now $(a, b) \in$ IIIa. Then by (3.7) we have $b > \frac{-1+\sqrt{5}}{2}r \Leftrightarrow b - r > \frac{3-\sqrt{5}}{2}b$. Now from Lemma 4.3, $1 + F_{n+1} > \frac{1+\sqrt{5}}{2}F_n \Rightarrow F_{n+2}\frac{3-\sqrt{5}}{2} > F_n$. From the assertion, $b > F_{n+2} \Leftrightarrow b\frac{3-\sqrt{5}}{2} > \frac{3-\sqrt{5}}{2}F_{n+2}$ and hence $b - r > F_n$ and, because $r'' = b - r$ by (3.9), we take $r'' > F_n$.

For 2. Let $(a, b) \in$ I. Then by (3.5) we have $r'' = b - r$ and $b - r < \frac{3-\sqrt{5}}{2}b$ and so, from the assertion, $r'' < \frac{3-\sqrt{5}}{2}\beta_{n+2}$ Finally, from Lemma 4.3, we take $r'' < \beta_{n+1}$. Let now $(a, b) \in$ IIIa. Then by (3.7) we have $r'' = b - r$ and $r'' < b/2$ and, by the assertion, $r'' < \beta_{n+2}/2$.

Lemma 4.3. *Let F_n be the Fibonacci sequence with $F_1 = F_2 = 1$ and $\beta_{n+2} = 2\beta_{n+1} + \beta_n$ with $\beta_1 = 2,\ \beta_2 = 5$. Then*
1. $\max(F_{n+3}, F_{n+2}) = n$,
$\max(F_{2n+4}, F_{2n+3}) = n + 1$,
$A(F_{2n+4}, F_{2n+3}) = n$.

2. $1 + F_{n+1} > \frac{-1+\sqrt{5}}{2}F_n$.

3. $\beta_n > \frac{-1+\sqrt{5}}{2}\beta_{n+1}$.

4. $\min(\beta_{n+1}, \beta_n) = n$

Proof. For 1. For $n = 1$, $\max(F_4, F_3) = \max(3, 2), \min(F_6, F_5) = \min(8, 5)$. We calculate the tree $< 8, 5 >$ which includes the tree $< 3, 2 >$.

$$\begin{tikzcd}
 & & & {[2,1]} \\
 & & {[3,2]} \arrow[ur] \arrow[dr] & \\
 & {[5,3]} \arrow[ur] \arrow[dr] & & {[2,1]} \\
{[8,5]} \arrow[ur] \arrow[ddr] & & {[3,1]} & \\
 & & {[2,1]} & \\
 & {[5,2]} \arrow[ur] \arrow[dr] & & \\
 & & {[2,1]} &
\end{tikzcd}$$

and so $\max(F_4, F_3) = 1 = n$ and $\min(F_6, F_5) = 2 = n + 1$.

We assume now that this is true for $n = k$. We will show that is true for $n = k + 1$ also. First we will show that $\frac{3}{2}F_{k+3} < F_{k+4} < 2F_{k+3}$ which means that $(F_{k+4}, F_{k+3}) \in \{$I, IIIa$\}$.
$\frac{3}{2}F_{k+3} < F_{k+4} \Leftrightarrow 3F_{k+3} < 2F_{k+3} + 2F_{k+2} \Leftrightarrow F_{k+2} + F_{k+1} < 2F_{k+2} \Leftrightarrow F_{k+1} < F_{k+2}$ and $F_{k+4} < 2F_{k+3} \Leftrightarrow F_{k+2} < F_{k+3}$. Thus we take the first step of the GEA for (F_{k+4}, F_{k+3}).

$$F_{k+4} = F_{k+3} + F_{k+2},\ F_{k+2} < F_{k+3}$$
$$F_{k+4} = 2F_{k+3} - F_{k+2},\ F_{k+1} < F_{k+3}$$

or

$$\begin{tikzcd}
 & {[F_{k+3}, F_{k+2}]} \\
{[F_{k+4}, F_{k+3}]} \arrow[ur] \arrow[dr] & \\
 & {[F_{k+3}, F_{k+1}]}
\end{tikzcd}$$

and because $(F_{k+4}, F_{k+3}) \in \{$I, IIIa$\}$we have from the Master Theorem that $\max(F_{k+4}, F_{k+3}) = \max(F_{k+3}, F_{k+2}) + 1$ and by the inductive hypothesis $= k + 1$.

Now for (F_{2k+6}, F_{2k+5}) we will show that $(F_{2k+6}, F_{2k+5}) \in$ IIIa. Equivalently we will show that

$$\frac{3}{2}F_{2k+5} < F_{2k+6} < F_{2k+5}\frac{1+\sqrt{5}}{2}$$

We will only prove that $F_{2k+6} < F_{2k+5}\frac{1+\sqrt{5}}{2}$. We use induction: $F_{2k+6} < F_{2k+5}\frac{1+\sqrt{5}}{2} \Leftrightarrow F_{2k+4}\frac{1+\sqrt{5}}{2} < F_{2k+5} \Leftrightarrow F_{2k+4}\frac{1+\sqrt{5}}{2} < F_{2k+4} + F_{2k+3} \Leftrightarrow F_{2k+4} < F_{2k+3}\frac{1+\sqrt{5}}{2}$ which is true by the inductive hypothesis. Thus we take the first step of GEA for the pair (F_{2k+6}, F_{2k+5}).

$$F_{2k+6} = F_{2k+5} + F_{2k+4},\ F_{2k+4} < F_{2k+5}$$
$$F_{2k+6} = 2F_{2k+5} - F_{2k+3},\ F_{2k+3} < F_{2k+5}$$

or

$$
\begin{tikzcd}[column sep=small]
 & {[F_{2k+5}, F_{2k+4}]} \\
{[F_{2k+6}, F_{2k+5}]} \arrow[ur] \arrow[dr] & \\
 & {[F_{2k+5}, F_{2k+3}]}
\end{tikzcd}
$$

and because $(F_{2k+6}, F_{2k+5}) \in$ IIIa we have from the Corollary 3.2 that $\min(F_{2k+6}, F_{2k+5}) = \min(F_{2k+5}, F_{2k+3}) + 1$. As now $< F_{2k+5}, F_{2k+3} >, < F_{2k+4}, F_{2k+3} >$ are equivalent trees ($F_{2k+5} \equiv F_{2k+4} \mod F_{2k+3}$) by Lemma 2.3 we take $\min(F_{2k+5}, F_{2k+3}) = \min(F_{2k+4}, F_{2k+3})$ and hence $\min(F_{2k+6}, F_{2k+5}) = \min(F_{2k+4}, F_{2k+3}) + 1$ and by the inductive hypothesis $= k + 2$.

Now we will show the third relation of part 1. of this lemma: $A(F_{2n+4}, F_{2n+3}) = \max(F_{2n+4}, F_{2n+3}) - \min(F_{2n+4}, F_{2n+3}) = (2n+1) - (n+1) = n$

For 2. For $n = 1$, $1 + F_2 > \frac{1+\sqrt{5}}{2} F_1 \Leftrightarrow 2 > \frac{1+\sqrt{5}}{2}$, which holds true. For $n = 2$, $1 + F_3 > \frac{1+\sqrt{5}}{2} F_2 \Leftrightarrow 3 > \frac{1+\sqrt{5}}{2}$ which is also true. Now we suppose that for every $n \leq k,\ 1 + F_{n+1} > \frac{1+\sqrt{5}}{2} F_n$ and we will show that $1 + F_{k+2} > \frac{1+\sqrt{5}}{2} F_{k+1}$. By the inductive hypothesis, $1 + F_k > \frac{1+\sqrt{5}}{2} F_{k-1}$ and as $\frac{3+\sqrt{5}}{2} > 1$, we get

$$
\begin{aligned}
& \frac{3+\sqrt{5}}{2} + F_k > \frac{1+\sqrt{5}}{2} F_{k-1} \\
\Leftrightarrow\ & \frac{-1+\sqrt{5}}{2} F_k + \frac{1+\sqrt{5}}{2} > F_{k-1} \\
\Leftrightarrow\ & \frac{1+\sqrt{5}}{2} F_k + \frac{1+\sqrt{5}}{2} > F_{k-1} + F_k = F_{k+1} \\
\Leftrightarrow\ & F_k + 1 > \frac{-1+\sqrt{5}}{2} F_{k+1} \\
\Leftrightarrow\ & F_{k+1} + F_k + 1 > \frac{1+\sqrt{5}}{2} F_{k+1} \\
\Leftrightarrow\ & 1 + F_{k+2} > \frac{1+\sqrt{5}}{2} F_{k+1}
\end{aligned}
$$

For 3. For $n = 1$ it holds, $\beta_1 > \frac{3-\sqrt{5}}{2} \beta_2 \Leftrightarrow 2 > \frac{3-\sqrt{5}}{2} 5 \Leftrightarrow 5\sqrt{5} > 11 \Leftrightarrow 125 > 121$. For $n = 2$, $\beta_2 > \frac{3-\sqrt{5}}{2} \beta_3 \Leftrightarrow 5 > \frac{3-\sqrt{5}}{2} 12 \Leftrightarrow 6\sqrt{5} > 13 \Leftrightarrow 180 > 169$, a valid statement.

Now, let the inductive hypothesis hold true for every $n \leq k$. Then

$$
\begin{aligned}
& \beta_{k-1} > \frac{3-\sqrt{5}}{2} \beta_k \\
& \beta_k > \frac{3-\sqrt{5}}{2} \beta_{k+1} \\
\Leftrightarrow\ & 2\beta_k > (3-\sqrt{5}) \beta_{k+1} \quad \textit{and by adding} \\
& \beta_{k-1} + 2\beta_k > \frac{3-\sqrt{5}}{2} (\beta_k + 2\beta_{k+1}) \\
\Leftrightarrow\ & \beta_{k+1} > \frac{3-\sqrt{5}}{2} \beta_{k+2}
\end{aligned}
$$

For 4. Like the first.

Theorem 4.4. *Let*
$$\Sigma'_n = \{A(c,d) : F_{2n+1} < d < c,\ gcd(c,d) = 1\}$$
where $A(c,d) = max(c,d) - min(c,d)$ *and* F_n *the* $Fibonacci\ sequence$*. Then* $\min \Sigma'_n = n$

Proof. It is clear that it is enough to show that $\min \Sigma_n = n$ where
$$\Sigma_n = \{A(c,d) : d < c,\ 1 + F_{2n+1} \leq d \leq F_{2n+3},\ gcd(c,d) = 1\}$$
For $n = 1$, we calculate $\Sigma_1 = \{A(4,3), A(5,3), A(5,4), A(7,4), A(6,5), A(7,5), A(8,5), A(9,5)\}$. As the pairs of trees $\{< 4,3 >, < 5,3 >\}$, $\{< 5,4 >, < 7,4 >\}$, $\{< 6,5 >, < 9,5 >\}$, $\{< 7,5 >, < 8,5 >\}$ are inverse, we will calculate only $A(5,3), A(5,4), A(8,5), A(6,5)$. By the first Tree of the proof of Lemma 4.3, we take $A(8,5) = \max(8,5) - \min(8,5) = 3 - 2 = 1$ and $A(5,3) = \max(5,3) - \min(5,3) = 2 - 1 = 1$. Now by

$$\begin{tikzcd}
 & {[4,1]} & & \\
{[5,4]} \arrow[ur] \arrow[dr] & & {[3,1]} & \\
 & {[4,3]} \arrow[ur] \arrow[dr] & & {[2,1]} \\
 & & {[3,2]} \arrow[ur] \arrow[dr] & \\
 & & & {[2,1]}
\end{tikzcd}$$

we have $A(5,4) = 2$ and similarly $A(6,5) = 3$. Then $\min \Sigma_1 = 1 = n$. Now we suppose that for every $n \leq k$ the theorem is true. We will show that $\min \Sigma_{k+1} = k + 1$.

Consider the tree $< a_1, a_2 >$ with
$$gcd(a_1, a_2) = 1 \tag{4.11}$$
Then there is $m \in \mathbb{N}$ such that
$$A(a_1, a_2) \in \Sigma_m \tag{4.12}$$
We assume also that
$$k + 1 = m \tag{4.13}$$
Now we choose a path from $< a_1, a_2 >$ by using the following procedure: The first vertex of the path is the root vertex $[a_1, a_2]$. Then if $(a_1, a_2) \in \{$ I, II $\}$ we choose the next vertex to be that with the greatest remainder of the two remainders which occur by a one-step application of the GEA for (a_1, a_2). Then from Lemma 4.1 we take $A(a_1, a_2) = A(a_2, a_3) + 1$ where a_3 is the greatest remainder. Now, if $(a_1, a_2) \in$ III, we choose the next vertex to be that with the smallest remainder, and hence by Lemma 4.1, $A(a_1, a_2) = A(a_2, a_3) + 1$, where a_3 is now the smallest remainder. We apply this procedure recursively and let the chosen path be
$$[a_1, a_2] \to [a_2, a_3] \to \ldots \to [a_\rho, 1]$$
Then we have $A(a_\lambda, a_{\lambda+1}) = A(a_{\lambda+1}, a_{\lambda+2}) + 1$ where $1 \leq \lambda \leq \rho - 1$ and $a_{\rho+1} = 1$. Then
$$A(a_1, a_2) = A(a_\sigma, a_{\sigma+1}) + \sigma - 1,\ 1 \leq \sigma \leq \rho \tag{4.14}$$

Now from (4.11) and by following the same arguments as in the Euclidean algorithm, we have

$$gcd(a_\sigma, a_{\sigma+1}) = 1,\ 1 \le \sigma \le \rho \tag{4.15}$$

Let now $\varrho \in \mathbb{N}$ be the greatest number such that

$$a_\varrho \ge 1 + F_{2k+3} \tag{4.16}$$

Then

$$a_{\varrho+1} \le F_{2k+3} \tag{4.17}$$

because if $a_{\varrho+1} > F_{2k+3}$ then $a_{\varrho+1} \ge 1 + F_{2k+3}$ and hence ϱ was not maximally chosen, a contradiction. From (4.16) and Lemma 4.2 we obtain

$$F_{2k+1} < a_{\varrho+1} \tag{4.18}$$

From (4.15) we have

$$gcd(a_\varrho, a_{\varrho+1}) = 1 \tag{4.19}$$

The relations (4.17),(4.18),(4.19) along with $a_\varrho > a_{\varrho+1}$ imply that $A(a_\varrho > a_{\varrho+1}) \in \Sigma_k$, and hence by the inductive hypothesis

$$A(a_\varrho, a_{\varrho+1}) \ge k \tag{4.20}$$

Now from (4.14) we have

$$A(a_1, a_2) = A(a_\varrho, a_{\varrho+1}) + \varrho - 1 \overset{(4.20)}{\ge} k + \varrho - 1$$

Thus

$$A(a_1, a_2) \ge k + \varrho - 1 \tag{4.21}$$

Let $\varrho = 1$. Then by (4.12), $A(a_\varrho, a_{\varrho+1}) \in \Sigma_m$, and by (4.13), $A(a_\varrho, a_{\varrho+1}) \in \Sigma_{k+1}$. So $1+F_{2k+3} \le a_{\varrho+1}$, which contradicts (4.17) and hence $\varrho \ge 2$ and along with (4.21) we obtain

$$A(a_1, a_2) \ge k + 1 \tag{4.22}$$

Finally we let $a_2 = F_{2k+5},\ a_1 = F_{2k+6}$. For these choises, we have by Lemma 4.3 that

$$A(a_1, a_2) = k + 1 \tag{4.23}$$

The relations (4.22),(4.23) give us $\min \Sigma_{k+1} = k + 1$, which completes the induction.

Another way to state Theorem 4.4 is the following: Let $\Lambda_n = \{d \in \mathbb{N} : A(c, d) = n,\ c < d,\ gcd(c, d) = 1\}$. Then $\max \Lambda_n = F_{2n+3}$.

Theorem 4.5. *Let*

$$B_n = \{\max(a, b) : 1 + F_{n+1} \le b \le F_{n+2},\ b < a,\ gcd(a, b) = 1\}$$

Then $\min B_n = n$

Proof. For $n = 1$, we have $B_1 = \{\max(3,2), \max(4,3), \max(5,3)\}$. As $< 4,3 >, < 5,3 >$ are inverse trees, we need only calculate $\max(3,2), \max(4,3)$. By the proof of Theorem 4.4, in the beginning, we have from the tree $< 5,4 >$ that $\max(3,2) = 1, \max(4,3) = 2$ and hence $\min B_1 = 1 = n$. Now we suppose that the theorem is true for every $n \leq k$ and we will show that $\min B_{k+1} = k+1$. $\square$

The proof of Theorem 4.5 is very similar to the proof of Theorem 4.4. We will only point out the differences.

Insted of (4.12), we have $\max(a_1, a_2) \in B_m$.

The chosen path from $< a_1, a_2 >$ is: The first vertex is the root vertex $[a_1, a_2]$. Now we choose the next vertex to be the one with the greatest remainder and we continue in the same way until we reach a leaf. Then, from Lemma 4.1, we have $\max(a_\lambda, a_{\lambda+1}) = \max(a_{\lambda+1}, a_{\lambda+2}) + 1$, $1 \leq \lambda \leq \rho - 1$. Then, $\max(a_1, a_2) = \max(a_\sigma, a_{\sigma+1}) + \sigma - 1, 1 \leq \sigma \leq \rho$. (instead of (4.14)).

Instead of (4.16) we have $a_\varrho \geq 1 + F_{k+2}$ and in the end we choose $a_1 = F_{k+4},\ a_2 = F_{k+3}$.

Another way to state Theorem 4.5 is the following: Let $A_n = \{b \in \mathbb{N} : \max(a,b) = n,\ a \in \mathbb{N},\ a > b,\ gcd(a,b) = 1\}$. Then $\max A_n = F_{n+2}$.

Theorem 4.6. *Let* $\beta_{n+2} = 2\beta_{n+1} + \beta_n$ *with* $\beta_1 = 2, \beta_2 = 5$ *and* $\Gamma_n = \{\min(a,b) : \beta_n \leq b < \beta_{n+1},\ b < a,\ gcd(a,b) = 1\}$. *Then* $\max \Gamma_n = n$.

Proof. For $n = 1$, $\Gamma_1 = \{\min(3,2), \min(4,3), \min(5,3), \min(5,4), \min(7,4)\}$ and again by $< 5,4 >$ we have $\min(3,2) = 1,\ \min(4,3) = \min(5,3) = 1, \min(4,3) = \min(5,3) = 1, \min(5,4) = \min(7,4) = 1$ and hence $\max \Gamma_1 = 1 = n$.

Now we suppose that for every $n \leq k$, the theorem is true and we will show that $\max \Gamma_{k+1} = k+1$.

Let (a_1, a_2) such that $\min(a_1, a_2) \in \Gamma_{k+1}$. Then

$$\beta_{k+1} \leq a_2 < \beta_{k+2} \tag{4.24}$$

$$gcd(a_1, a_2) = 1 \tag{4.25}$$

and $a_2 < a_1$.

Now we consider the case where $(a_1, a_2) \in$ I. Then by Corollary 3.3 we obtain

$$\min(a_1, a_2) = \min(a_2, a_3) + 1 \tag{4.26}$$

where a_3 is the smaller remainder between the two taken by applying one time the GEA for (a_1, a_2). Also by (4.24) and Lemma 4.2 it holds that

$$a_3 < \beta_{k+1} \tag{4.27}$$

which, by the inductive hypothesis, means that $\min(a_2, a_3) \in \Gamma_\lambda$ where $\lambda \leq k$. Thus by (4.26) we have $\min(a_1, a_2) \leq k+1$. Now by Lemma 4.3 for $a_1 = \beta_{k+2},\ a_2 = \beta_{k+1}$ we have $\min(a_1, a_2) = k+1$ and hence $\max \Gamma_{k+1} = k+1$.

Let now $(a_1, a_2) \in$ IIIa. Then, by Corollary 3.3,

$$\min(a_1, a_2) = \min(a_2, a_3) + 1 \tag{4.28}$$

where a_3 is the smaller remainder.
By (4.24) and Lemma 4.2

$$a_3 < \beta_{k+2}/2 \tag{4.29}$$

By (3.7)

$$a_3 < a_2/2 \tag{4.30}$$

By (4.29), we can consider two cases. If $a_3 < \beta_{k+1}$ then we follow the same arguments as in the case $(a_1, a_2) \in$ I after (4.27).

Now let

$$\beta_{k+1} \leq a_3 \tag{4.31}$$

Then by (4.24),(4.31),(4.30) $a_2 > 2a_3 \geq 2\beta_{k+1}$ and hence

$$2\beta_{k+1} < a_2 < \beta_{k+2} \tag{4.32}$$

Now we will calculate the smaller remainder which occurs in the first step of GEA for (a_2, a_3). First we see that, by (4.32) and (4.31)

$$a_2 < \beta_{k+2} = 2\beta_{k+1} + \beta_k < 3\beta_{k+1} \leq 3a_3$$

and hence by (4.30)

$$2a_3 < a_2 < 3a_3$$

which means that the smaller remainder a_4 is either $a_2 - 2a_3$ or $3a_3 - a_2$ and to find it, we use relations (4.31),(4.32):

$$\begin{aligned} & 2\beta_k < \beta_{k+1} \\ \Leftrightarrow \quad & 4\beta_{k+1} + 2\beta_k < 5\beta_{k+1} \\ \Leftrightarrow \quad & 2\beta_{k+2} < 5\beta_{k+1} \\ \Rightarrow \quad & 2a_2 < 5a_3 \\ \Leftrightarrow \quad & a_2 - 2a_3 < 3a_3 - a_2 \end{aligned}$$

and hence $a_4 = a_2 - 2a_3$. By (4.31),(4.32) we obtain $a_4 < \beta_k$ and thus, by the inductive hypothesis, $\min(a_3, a_4) \leq k-1$. Finally by Corollary 3.3 we have $\min(a_2, a_3) = \min(a_3, a_4)+1$ and so by (4.28) we have

$$\min(a_1, a_2) = \min(a_3, a_4) + 2 \leq (k-1) + 2 = k + 1$$

and hence for $a_1 = \beta_{k+2},\ a_2 = \beta_{k+1}$ by Lemma 4.3 we have $\min(a_1, a_2) = k + 1$ and so $\max \Gamma_{k+1} = k + 1$ in case $(a_1, a_2) \in$ IIIa.

We treat cases II,IIIb similarly. $\square$

Another way to state Theorem 4.6 is: Let $\Delta_n = \{b \in \mathbb{N} : \min(a, b) = n,\ a > b,\ gcd(a, b) = 1\}$. Then $\min \Delta_n = \beta_n$.

5 Two criteria for finding the SP of the Euclidean tree based on the geometric and the harmonic mean

In this chapter, we will prove two criteria, for finding a SP of the Euclidean Tree, resulting from the Harmonic and the Geometric Mean.

The theorem of Vahlen-Kronecker [3] for the GEA for $< a, b >$, states that: If we choose the children with the smaller remainder(or equal sometimes at the last step of the algorithm) at every step, then we take a SP of $< a, b >$.

Equivalently, by letting $r, b-r$ ($r = a - qb$) be the two remainders of the first step of the GEA for (a, b), if $b - r < r$ then $[b, b-r]$ is in SP of $< a, b >$ and if $b - r > r$ then $[b, r]$ is in SP of $< a, b >$.

Again, equivalently, by replacing $r = a - qb$ we have, if $a > \frac{qb+(q+1)b}{2}$ then $[b, b-r]$ is in SP of $< a, b >$, and if $a < \frac{qb+(q+1)b}{2}$ then $[b, r]$ is in SP of $< a, b >$.

If now, instead of the arithmetic mean of qb and $(q+1)b$, we consider the geometric mean, we get the same result and almost the same happens in the case we consider the harmonic mean.

Theorem 5.1. *Let $< a, b >$ with (1.1)(1.2): $qb < a < (q+1)b$, $r = a - qb$. Then*
1. If $a > b\sqrt{q(q+1)}$, then $[b, b-r]$ is in SP of $< a, b >$ and if $a < b\sqrt{q(q+1)}$, then $[b, r]$ is in SP of of $< a, b >$.
2. If $a > \frac{2bq(q+1)}{2q+1}$ and $q \geq 2$, then $[b, b-r]$ is in SP of $< a, b >$, and if $a < \frac{2bq(q+1)}{2q+1}$, then $[b, r]$ is in SP of $< a, b >$.

Proof. For 1. Let $a > b\sqrt{q(q+1)}$. To show that $[b, b-r]$ is in SP of $< a, b >$, it is enough to show that, by the Master theorem, $(a, b) \in \{\text{I, III}\}$ and equivalently $b(q + \frac{3-\sqrt{5}}{2}) < a$. Thus by the assertion, it is enough to show that $q + \frac{3-\sqrt{5}}{2} < \sqrt{q(q+1)}$. Let $J(q) = q + \frac{3-\sqrt{5}}{2} - \sqrt{q(q+1)}$. Then $J'(q) = 1 - \frac{2q+1}{\sqrt{q(q+1)}} < 0$ and so J is strictly decreasing. Thus it remains to show that $J(1) < 0$. We calculate $J(1) = 1 - \sqrt{2} + \frac{3-\sqrt{5}}{2} < 0$.

Now let $a < b\sqrt{q(q+1)}$. Then, because the geometric mean is less than the arithmetic mean, we obtain $a < b(q + \frac{1}{2})$. Hence, by the Master Theorem, we have what we required.

For 2. Let $a < \frac{2bq(q+1)}{2q+1}$. Then, because the harmonic mean is less than the arithmetic mean, we argue as in the previous paragraph and obtain the result.

Now let $a > \frac{2bq(q+1)}{2q+1}$, $q \geq 2$. To show that $[b, b-r]$ is in SP of $< a, b >$, it is enough to show that, by the Master Theorem, that $(a, b) \in \{\text{I, III}\}$ and equivalently $b(q - \frac{3-\sqrt{5}}{2}) < a$. thus by the assertion, it is enough to show that $q + \frac{3-\sqrt{5}}{2} < \frac{2q(q+1)}{2q+1}$ for $q \geq 2$. Let $J(q) = (2q+1)(q + \frac{3-\sqrt{5}}{2}) - 2q(q+1)$. Then $J'(q) = -2 + \sqrt{5} > 0$ which means that J is strictly increasing and because $J(2) = \frac{-11+\sqrt{5}}{2} > 0$ we have what we required.

Remarks.

1. We can combine the Vahlen-Kronecker criterion with the harmonic mean: If

$$\frac{2bq(q+1)}{2q+1} < a < b(q + \frac{1}{2}),\ q \geq 2$$

then $(a, b) \in$ IIIb and so, both vertices are in SP of $< a, b >$.

And in the case $b(q+\frac{1}{2}) < a,\ q \geq 2$ we consider the inverse tree $< a', b >$ for which we have, $a' < b(q+\frac{1}{2})$ and Lemma 2.5 and we check if $\frac{2bq(q+1)}{2q+1} < a'$.
2. We can surpass the condition $q \geq 2$ of the harmonic criterion by considering $< a+b, b >$, the equivalent tree of $< a, b >$ because of the Lemma 2.3 and because the quotient of the first step of GEA for $(a+b, b)$ is ≥ 2 and hence we apply the criterion for $(a+b, b)$ instead of (a, b).

6 A relation between two specific paths of the Euclidean tree

Here we construct an algorithm(NPA) which is based on the first theorem of [2] which presents a connection between the Least Remainder Algorithm and the Euclidean Algorithm. We will prove a similar theorem which connects EA with this new algorithm.

Consider $< a, b >$ with (1.1)(1.2): $qb < a < (q+1)b$, $r = a - qb$. Then we have (1.4) :

$$a = qb + r,\ 0 < r < b$$
$$a = (q+1)b - (b-r),\ 0 < b - r < b$$

From now on we will call r the positive remainder of the first step of GEA for (a, b) and $b - r$ the negative one. In the Euclidean algorithm we choose every time the vertex with the positive remainder. Now we define the **Least Remainder Algorithm (LRA)**: If $(a, b) \in \{$ I, IIIa $\}$ we choose the next vertex to be that with the negative remainder ($[b, b-r]$) and in case $(a, b) \in \{$II, IIIb$\}$ we choose that with the positive one ($[b, r]$) and we continue this process until we reach a leaf of $< a, b >$.

Theorem I of [2] states that, for a given pair (a, b), the number of vertices of the LRA path(except the root vertex) which have negative remainder equals to the number of steps of the Euclidean Algorithm minus the number of steps of LRA.

Now we will construct an algorithm for which we will prove a similar theorem. **Algorithm NP** : If $(a, b) \in \{$ I, IIIa $\}$ we choose the next vertex to be that with the negative remainder ($[b, b-r]$). In case $(a, b) \in \{$ II, IIIb $\}$ we choose the next two vertices. The first is that with the negative remainder and the next one that with the positive remainder. And we continue this process until we reach a leaf of $< a, b >$.

According to the Master theorem, LRA gives a SP of $< a, b >$.

Theorem 6.1. *For a given pair (a, b), the number of vertices of the LRA path(except the root vertex) which have positive remainder equals to the number of steps of the NP Algorithm minus the number of steps of LRA.*

Proof. By the definition of LRA and NP algorithm when $(a, b) \in \{$I, IIIa$\}$, the two algorithms are identical and they both follow the vertex with the negative remainder.

Let now $(a, b) \in \{$II, IIIb$\}$ with (1.1)(1.2): $qb < a < (q+1)b$, $r = a - qb$. In this case we have $a < b(q+\frac{1}{2})$. By replacing $a = qb + r$ we take $0 < r < b/2$ and equivalently

$$b - r < b < 2(b-r)$$

which give us the first step of GEA for the pair $(b, b-r)$:

$$
\begin{aligned}
b &= 1 \cdot (b-r) + r,\ 0 < r < b-r \\
b &= 2 \cdot (b-r) - (b-2r),\ 0 < b-2r < b-r
\end{aligned}
$$

or

$$
\begin{tikzcd}
& {[b,r]} & \\
{[a,b]} \arrow[ur] \arrow[dr] & & {[b-r,r]} \\
& {[b,b-r]} \arrow[ur] \arrow[dr] & \\
& & {[b-r,b-2r]}
\end{tikzcd}
$$

and equivalently by the definition of LRA and NP algorithm

$$
\begin{tikzcd}
& {[b,r]} & \\
{[a,b]} \arrow[ur, "LRA", "+"'] \arrow[dr, "NP", "-"'] & & {[b-r,r]} \\
& {[b,b-r]} \arrow[ur, "NP", "+"'] \arrow[dr, "-"] & \\
& & {[b-r,b-2r]}
\end{tikzcd}
$$

As now $b \equiv (b-r) \mod r$, $< b, r >$,$< b-r, r >$ are equivalent trees and thus, by Lemma 2.3, are identical except for the root vertex. Because also we have that the two algorithms are identical in case $(a, b) \in \{$I, IIIa$\}$and follow the vertex with the negative remainder, it is clear that for every vertex of the LRA path which have positive remainder corresponds an additional vertex for NP path. And the sum of these additional vertices is the difference between the number of steps of NP algorithm and LRA.

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