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On s_k–Jacobsthal numbers

Aldous Cesar F. Bueno

Philippine Science High Scool-Central Luzon Campus
Clark Freeport Zone, Angeles City, Pampanga, Philippines
e-mail: aldouz_cezar@yahoo.com

Abstract: In this paper the s_k–Jacobsthal numbers are introduced and their properties are studied.
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1 Introduction

The n-th Jacobsthal number ($n \geq 0$) is given by

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{1}$$

There have been modifications on this equation that were shown in [1]. Now we give a new modification and the resulting numbers will be called the s_k–Jacobsthal numbers.

The n^{th} s_k–Jacobsthal number is given by

$$J_n^{s_k} = \frac{s_k^n - (-1)^n}{s_{k+1}}, \tag{2}$$

where $s_k \neq 0$ and $s_{k+1} \neq 0, 1$ are the k^{th} and $(k+1)^{th}$ terms of the real sequence $\{s_k\}_{k=0}^{+\infty}$.

2 Main results

Theorem 2.1.

$$J_{n+1}^{s_k} = (s_k - 1)J_n^{s_k} + s_k J_{n-1}^{s_k}. \tag{3}$$

Proof:

$$J_{n+1}^{s_k} = \frac{s_k^{n+1} - (-1)^{n+1}}{s_{k+1}}.$$

Note that $s_{k+1}^{n+1} = s_k s_{k+1} J_n^{s_k}$ and $(-1)^{n+1} = s_{k+1} J_n^{s_k} - s_k^n$.

Substituting these values gives

$$\begin{aligned}
J_{n+1}^{s_k} &= \frac{s_k s_{k+1} J_n^{s_k} + (-1)^n s_k - s_{k+1} J_n^{s_k} + s_k^n}{s_{k+1}} \\
&= \frac{s_k s_{k+1} J_n^{s_k} - s_{k+1} J_n^{s_k} + s_k^n - (-1)^{n-1} s_k}{s_{k+1}} \\
&= (s_k - 1) J_n^{s_k} + \frac{s_k [s_k^{n-1} - (-1)^{n-1}]}{s_{k+1}} \\
&= (s_k - 1) J_n^{s_k} + s_k J_{n-1}^{s_k}.
\end{aligned}$$

Theorem 2.2.

$$J_n^{s_k} J_{n+1}^{s_k} = \left[\frac{2 s_k^{2n+1}}{s_{k+1}} - J_{2n+1}^{s_k}\right] \left[\frac{2(-s_k)^n}{s_{k+1}} - (-s_k)^n J_1^{s_k}\right]. \tag{4}$$

Proof:

$$\begin{aligned}
J_n^{s_k} J_{n+1}^{s_k} &= \frac{(s_k^n - (-1)^n)(s_k^{n+1} - (-1)^{n+1})}{s_{k+1}^2} \\
&= \frac{s_k^{2n+1} - (-1)^n s_k^{n+1} + (-1)^n s_k^n + (-1)^{2n+1}}{s_{k+1}^2} \\
&= \left[\frac{s_k^{2n+1} + (-1)^{2n+1}}{s_{k+1}}\right] \left[\frac{-(-s_k)^n [s_k - 1]}{s_{k+1}}\right] \\
&= \left[\frac{2 s_k^{2n+1}}{s_{k+1}} - J_{2n+1}^{s_k}\right] \left[\frac{2(-s_k)^n}{s_{k+1}} - (-s_k)^n J_1^{s_k}\right]
\end{aligned}$$

Theorem 2.3.

$$J_n^{s_k} + J_{n+1}^{s_k} = s_k^n J_1^{s_k} \tag{5}$$

Proof:

$$\begin{aligned}
J_n^{s_k} + J_{n+1}^{s_k} &= \frac{s_k^n - (-1)^n}{s_{k+1}} + \frac{s_k^{n+1} - (-1)^{n+1}}{s_{k+1}} \\
&= \frac{s_k^{n+1} + s_k^n}{s_{k+1}} \\
&= \frac{s_k^n (s_k + 1)}{s_{k+1}} \\
&= s_k^n J_1^{s_k}
\end{aligned}$$

Theorem 2.4.

$$\sum_{m=0}^{n-1} J_m^{s_k} = \frac{1}{2} J_{n-1}^{s_k} + \frac{s_k^{n-1} - 1}{2 s_{k+1}} + \frac{1 - s_k^{n-1}}{2 - s_{k+1} J_1^{s_k}} \tag{6}$$

Proof:

$$\begin{aligned}
\sum_{m=0}^{n-1} J_m^{s_k} &= \sum_{m=0}^{n-1} \left[\frac{s_k^m - (-1)^m}{s_{k+1}}\right] \\
&= \frac{1}{s_{k+1}} \sum_{m=0}^{n-1} s_k^m - \frac{1 - (-1)^n}{2 s_{k+1}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{s_k^{n-1}-(-1)^{n-1}}{s_{k+1}} + \frac{1}{s_{k+1}}\sum_{m=0}^{n-2} s_k^m + \frac{s_k^{n-1}-s_{k+1}J_{n-1}^{s_l}}{2s_{k+1}} - \frac{1}{2s_{k+1}} \\
&= J_{n-1}^{s_k} + \frac{s_k^{n-1}}{2s_{k+1}} - \frac{1}{2}J_{n-1}^{s_k} - \frac{1}{2s_{k+1}} + \frac{1-s_k^{n-1}}{1-s_k} \\
&= = \frac{1}{2}J_{n-1}^{s_k} + \frac{s_k^{n-1}-1}{2s_{k+1}} + \frac{1-s_k^{n-1}}{2-s_{k+1}J_1^{s_k}}
\end{aligned}$$

3 Examples

Example 3.1. *If* $\{s_k\}_{k=0}^{+\infty} = \{F_k\}_{k=0}^{+\infty}$ *we have the following*

1. $J_n^{F_k} = \frac{F_k^n-(-1)^n}{F_{k+1}}$
2. $J_{n+1}^{F_k} = (F_k-1)J_k^{F_k} + F_k J_{n-1}^{F_k}$
3. $J_n^{F_k} J_{n+1}^{F_k} = \left[\frac{2F_k^{2n+1}}{F_{k+1}} - J_{2n+1}^{F_k}\right]\left[\frac{2(-F_k)^n}{F_{k+1}} - (-F_k)^n J_1^{F_k}\right]$
4. $J_n^{F_k} + J_{n+1}^{F_k} = F_k^n J_1^{F_k}$
5. $\sum_{m=0}^{n-1} J_m^{F_k} = \frac{1}{2}J_{n-1}^{F_k} + \frac{F_k^{n-1}-1}{2F_{k+1}} + \frac{1-F_k^{n-1}}{2-F_{k+1}J_1^{F_k}}$

Remarks:

- If we choose $s_k = F_2 = 2$ and $s_{k+1} = F_3 = 3$, we have $J_n^{F_2} = J_n$.
- It is also easy to show that
$$\lim_{n\to+\infty} \frac{J_{n+1}^{F_k}}{J_n^{F_k}} = F_k$$

Example 3.2. *For* $\{s_k\}_{k=0}^{+\infty} = \left\{2^k\right\}_{k=0}^{+\infty}$ *we have the following results*

1. $J_n^{2^k} = \frac{2^{kn}-(-1)^n}{2^{k+1}}$
2. $J_{n+1}^{2^k} = (2^k-1)J_n^{2^k} + 2^k J_{n-1}^{2^k}$
3. $J_n^{2^k} J_{n+1}^{2^k} = \left[\frac{2^{2kn+k+1}}{2^{k+1}} - J_{2n+1}^{2^k}\right]\left[\frac{2(-2^k)^n}{2^{k+1}-(-2^k)^n J_1^{2^k}}\right]$
4. $J_n^{2^k} + J_{n+1}^{2^k} = 2^{kn} J_1^{2^k}$
5. $\sum_{k=0}^{n-1} J_m^{2^k} = \frac{1}{2}J_{n-1}^{2^k} + \frac{2^{kn-k}-1}{2^{k+2}} - \frac{1-2^{kn-k}}{2-2^{k+1}J_1^{2^k}}$

References

[1] Atanassov, K. Short remarks on Jacobsthal numbers, *Notes on Number Theory and Discrete Mathematics*, Vol. 18, 2012, No. 2, 63–64.