Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 20, 2014, No. 2, 36–44

On summation of certain infinite series and sum of powers of square root of natural numbers

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Abstract: Summation of certain infinite series involving powers of square root of natural numbers is evaluated through Riemann zeta function. The sum of powers of square root of first n natural numbers are expressed in terms of infinite series and Riemann zeta function.
Keywords: Hurwitz zeta function, Riemann zeta function, Infinite series, Ramanujan notebooks.
AMS Classification: 11M06, 11M35, 40A05.

1 Introduction

Ramanujan studied summation of following infinite series (1.1) for $x \geq 0$ and odd integer s in his paper, titled "Summation of a certain series", through Hurwitz zeta function and Riemann zeta function [3].

$$\begin{aligned}\Phi(x,s) &= \sum_{k=0}^{\infty} \frac{1}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^s} \qquad (1.1)\\ &= \sum_{k=0}^{\infty} \left(\sqrt{x+k+1}-\sqrt{x+k}\right)^s .\end{aligned}$$

$\Phi(x,s)$ and $\Phi(0,s)$ can be expressed in terms of Hurwitz zeta function $\zeta(x,s)$ and Riemann zeta function $\zeta(s)$ respectively. In [4], he gave some finite expressions for sum of powers of n natural numbers in terms of zeta function and Φ. Some of them are

$$\sqrt{1}+\sqrt{2}+\ldots+\sqrt{n} = \frac{2}{3}n^{\frac{3}{2}} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \frac{1}{6}\Phi(3,n). \qquad (1.2)$$

$$1^{\frac{3}{2}} + 2^{\frac{3}{2}} + \ldots + n^{\frac{3}{2}} = \frac{2n^{\frac{5}{2}}}{5} + \frac{n^{\frac{3}{2}}}{2} + \frac{\sqrt{n}}{8} - \frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right) + \frac{\Phi(5,n)}{40}. \qquad (1.3)$$

Further, he pointed that the higher powers such as $\sum_{k=1}^{n} k^{5/2}$, $\sum_{k=1}^{n} k^{7/2}, \ldots$ are not so neat as (1.2) and (1.3). Similar expressions for sum of powers of square root of first n natural numbers are also found in [1, p.455-457].

In the present work, we study the summation of following infinite series for odd integer s greater than $2n+2$ and $n \in N$.

$$\sum_{k=0}^{\infty} \frac{\left(\sqrt{(x+k)(x+k+1)}\right)^n}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^s}.$$

We give some finite expressions for sum of powers of first n natural numbers in terms of above mentioned infinite series and Riemann zeta function. Moreover, we give some elegant expressions for $\sum_{k=1}^{n} k^{5/2}$ and $\sum_{k=1}^{n} k^{7/2}$ in terms of single infinite series and Riemann zeta function as given in (1.2) and (1.3).

2 Preliminaries

Definition 2.1. For $n \in N$, $x \geq 0$ and s be an odd integer greater than $2n+2$, then define

$$\Phi(n,x,s) = \sum_{k=0}^{\infty} \frac{\left(\sqrt{(x+k)(x+k+1)}\right)^n}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^s}. \tag{2.1}$$

From the definition 2.1, it is clear that

$$\Phi(n,0,s) = \begin{cases} \Phi(n,1,s) & \text{If } n \in N \\ 1+\Phi(n,1,s) & \text{If } n = 0. \end{cases} \tag{2.2}$$

Usually, the Hurwitz zeta function and Riemann zeta function [2, p.1027] are defined by

$$\zeta(s,x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^s} \quad \text{and} \quad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

respectively. The functional equation satisfied by $\zeta(s)$ [2, p. 1028], viz

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\left(\frac{\pi s}{2}\right). \tag{2.3}$$

Further, we have following identities from Ref [3]

$$\begin{cases} 1^s + 2^s + \ldots + n^s = \zeta(-s) - \zeta(-s, n+1). \\ 1^s + 3^s + \ldots + (2n-1)^s = (1-2^s)\zeta(-s) - 2^s\zeta(-s, n+1). \end{cases} \tag{2.4}$$

Let

$$f(t,a,b) = \left(a+b\sqrt{t}\right)^s + \left(a-b\sqrt{t}\right)^s. \tag{2.5}$$

Then, expand (2.5) by binomial theorem and after simplification, gives

$$2\sum_{m=k}^{\frac{s-1}{2}} \binom{s}{2m}\binom{m}{k} a^{s-2m}b^{2m} = \frac{1}{k!}\,\frac{\partial^k}{\partial t^k}\left[f(t,a,b)\right]\Big|_{t=1}. \tag{2.6}$$

3 Main results

Theorem 3.1. *If $s > 2m+2$ and $m \in W$, then $\Phi(m,x,s)$ converges for all $x \geq 0$.*

Proof. From the definition 2.1

$$\Phi(m,x,s) = \sum_{k=0}^{\infty} \frac{\left(\sqrt{(x+k)(x+k+1)}\right)^m}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^s}.$$

Using the inequality $x+k < x+k+1$, $x \geq 0$ and $k \in W$, gives

$$\Phi(m,x,s) < \frac{1}{2^s}\sum_{k=0}^{\infty} \frac{(x+k+1)^m}{(x+k)^{\frac{s}{2}}}.$$

Using binomial theorem and definition of Hurwitz zeta function, yields

$$\Phi(m,x,s) < \frac{1}{2^s}\sum_{n=0}^{m}\binom{m}{n}\zeta\left(\frac{s}{2}-n,x\right). \tag{3.1}$$

The right hand side of (3.1) exists if $\frac{s}{2}-m > 1$. Hence, $\Phi(m,x,s)$ converges if $s > 2m+2$. □

Theorem 3.2. *Let $m \in N$ and s be an odd integer. If $s > 4m+2$, then*

$$\begin{aligned}\Phi(m,x,s-2m) = &-\frac{1}{2}\left(x^2-x\right)^{\frac{m}{2}} g(x,s,m) \\ &+(-1)^{m+1}\frac{(s-2m)!}{s!}\sum_{k=1,2\nmid k}^{(s-1)/2}\zeta(k-s/2,x)\frac{f_m^{(k)}}{k!}.\end{aligned} \tag{3.2}$$

where

$$f_m^{(k)} = \frac{\partial^{2m+k}}{\partial a^m \partial b^m \partial x^k}\left[f(a,b,x)\right]\Big|_{a=1,b=1,x=1} \tag{3.3}$$

$$g(x,s,m) = (-1)^m\left(\sqrt{x}+\sqrt{x-1}\right)^{s-2m} + \left(\sqrt{x}-\sqrt{x-1}\right)^{s-2m}. \tag{3.4}$$

Proof. For $a, b \in R$ and $x \geq 0$, define

$$\begin{aligned}\gamma(a,b,x) = &\left(a\sqrt{x}+b\sqrt{x+1}\right)^s + \left(a\sqrt{x}-b\sqrt{x+1}\right)^s \\ &-\left(a\sqrt{x}+b\sqrt{x-1}\right)^s - \left(a\sqrt{x}-b\sqrt{x-1}\right)^s.\end{aligned} \tag{3.5}$$

Using binomial theorem, expand right hand of (3.5) in ascending powers of $\sqrt{x}$ and after simplification, that

$$\gamma(a,b,x) = 4 \sum_{k=1,2\nmid k}^{\frac{s-1}{2}} x^{s/2-k} \sum_{m=k}^{\frac{s-1}{2}} \binom{s}{2m} \binom{m}{k} a^{s-2m} b^{2m}. \tag{3.6}$$

Using (2.6) in (3.5) can be written as

$$\gamma(a,b,x) = 2 \sum_{k=1,2\nmid k}^{\frac{s-1}{2}} \frac{1}{k!} x^{s/2-k} \left.\frac{\partial^k f}{\partial t^k}\right|_{t=1}. \tag{3.7}$$

Replacing x by $x+n$ and taking summation on both sides for $n = 0, 1, 2, \ldots$ and after simplification, gives

$$\sum_{n=0}^{\infty} \gamma(a,b,x+n) = 2 \sum_{k=1,2\nmid k}^{\frac{s-1}{2}} \frac{1}{k!} \zeta(k-s/2,x) \left.\frac{\partial^k f}{\partial t^k}\right|_{t=1}. \tag{3.8}$$

Differentiate (3.8) m times partially with respect to a and b, setting $a=1$ and $b=1$, then

$$\sum_{n=0}^{\infty} \left.\frac{\partial^{2m}\left[\gamma(a,b,x+n)\right]}{\partial a^m \partial b^m}\right|_{a=1,b=1} = 2 \sum_{k=1,2\nmid k}^{\frac{s-1}{2}} \frac{f_m^{(k)}}{k!} \zeta(k-s/2,x). \tag{3.9}$$

where $f_m^{(k)} = \frac{\partial^{2m+k}}{\partial a^m \partial b^m \partial x^k}\left[f(a,b,x)\right]\Big|_{a=1,b=1,x=1}$. Now, differentiate (3.5) m times partially with respect to a and b, set $a=1$ and $b=1$, then

$$\begin{aligned}
\frac{\partial^{2m}\gamma(x,a,b)}{\partial a^m \partial b^m} \frac{(s-2m)!}{s!} &= \left(a\sqrt{x}+b\sqrt{x+1}\right)^{s-2m} (\sqrt{x}\sqrt{x+1})^m \\
&\quad +(-1)^m \left(a\sqrt{x}-b\sqrt{x+1}\right)^{s-2m} (\sqrt{x}\sqrt{x+1})^m \\
&\quad - \left(a\sqrt{x}+b\sqrt{x-1}\right)^{s-2m} (\sqrt{x}\sqrt{x-1})^m \\
&\quad -(-1)^m \left(a\sqrt{x}-b\sqrt{x-1}\right)^{s-2m} (\sqrt{x}\sqrt{x-1})^m.
\end{aligned}$$

Replace x by $x+n$ and then taking summation on both sides for $n = 0, 1, 2, \ldots$ and using (2.1), gives

$$\begin{aligned}
\frac{(s-2m)!}{s!} \sum_{n=0}^{\infty} \left.\frac{\partial^{2m}\gamma(x+n,a,b)}{\partial a^m \partial b^m}\right|_{a=1,b=1} &= 2(-1)^{m+1}\phi(m,x,s-2m) \\
-\left(x^2-x\right)^{m/2} \left(\left[\sqrt{x}+\sqrt{x-1}\right]^{s-2m} + (-1)^m \left[\sqrt{x}-\sqrt{x-1}\right]^{s-2m}\right).
\end{aligned}$$

Using (3.8) in above equation and after simplification, gives (3.2). □

Theorem 3.3. *If $m \in N$ and $s > 2m+1$ is an odd integer, then*

$$\Phi(m,0,s-2m) = 2(-1)^{m+1}\frac{(s-2m)!}{s!}\sum_{k=1,2\nmid k}^{(s-1)/2}\frac{f_m^{(k)}}{k!}A(k). \tag{3.10}$$

where

$$A(k) = (-1)^{\frac{k-1}{2}}(2\pi)^{k-\frac{s}{2}-1}\Gamma\left(1-k+\frac{s}{2}\right)\zeta\left(1-k+\frac{s}{2}\right)\cos\frac{s\pi}{4}. \tag{3.11}$$

Proof. Let $x=1$ in (3.1). then

$$\Phi(m,0,s-2m) = (-1)^{m+1}\frac{(s-2m)!}{s!}\sum_{k=1,2\nmid k}^{(s-1)/2}\frac{f_m^{(k)}}{k!}\zeta(k-s/2).$$

Using (2.3) and after simplification, gives (3.10). □

Theorem 3.4. *If $m \in N$ and $s > 4m+2$ is an odd integer, then*

$$\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\sqrt{2k+1}}+\frac{1}{\sqrt{2k+3}}\right)^{-m}}{\left(\sqrt{2k+1}+\sqrt{2k+3}\right)^{s-2m}} = -\frac{C(m)}{2}\left(\frac{1}{\sqrt{2}}\right)^{s} \tag{3.12}$$
$$+2(-1)^{m+1}\frac{(s-2m)!}{s!}\sum_{k=1,2\nmid k}^{\frac{(s-1)}{2}}\left(1-2^{k-\frac{s}{2}}\right)\frac{f_m^{(k)}}{k!}A(k).$$

where

$$C(m) = i^m\left[(-1)^m(1+i)^{s-2m}+(1-i)^{s-2m}\right]. \tag{3.13}$$

Proof. Put $x=1/2$ in (3.1) and simplification, yields (3.12). □

Theorem 3.5. *If $n \in N$ and $s > 4m+2$ is an odd integer, then*

$$\sum_{k=1,2\nmid k}^{\frac{(s-1)}{2}}\frac{f_m^{(k)}}{k!}S_{\frac{s}{2}-k}(n) = \sum_{k=1,2\nmid k}^{\frac{(s-1)}{2}}\frac{f_m^{(k)}}{k!}\left(2A(k)+n^{\frac{s}{2}-k}\right) \tag{3.14}$$
$$+\frac{(-1)^m s!}{(s-2m)!}\left[\phi(m,n,s-2m)+\frac{1}{2}\left(n^2-n\right)^{\frac{m}{2}}g(n,s,m)\right].$$

Where $S_m(n) = \sum_{k=1}^{n}k^m$.

Proof. Put $x = n$ in (3.2), then

$$\Phi(m, n, s-2m) = -\frac{1}{2}\left(n^2 - n\right)^{\frac{m}{2}} g(n, s, m)$$
$$+(-1)^{m+1}\frac{(s-2m)!}{s!}\sum_{k=1, 2\nmid k}^{(s-1)/2} \zeta(k - s/2, n)\frac{f_m^{(k)}}{k!}.$$

Using (2.4) and after simplification, we find

$$\left[\Phi(m, n, s-2m) + \frac{1}{2}\left(n^2 - n\right)^{\frac{m}{2}} g(n, s, m)\right]\frac{(-1)^{m+1}s!}{(s-2m)!}$$
$$= \sum_{k=1,,2\nmid k}^{(s-1)/2} \left(2A(k) - S_{s/2-k}(n) + n^{s/2-k}\right)\frac{f_m^{(k)}}{k!}.$$

After simplification, completes the proof. □

4 Evaluation of certain infinite series

4.1 Summation of infinite series through main results

In this subsection, some infinite series are expressed in finite terms using theorems in section 3.

Example 4.1. Let $m = 1$ and $s = 7, 9$ in (3.2). Then

$$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right)^{-1}}{\left(\sqrt{k} + \sqrt{k+1}\right)^4} = \frac{105}{8\pi^3}\zeta\left(\frac{7}{2}\right) - \frac{1}{2\pi}\zeta\left(\frac{3}{2}\right).$$
$$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right)^{-1}}{\left(\sqrt{k} + \sqrt{k+1}\right)^6} = \frac{945}{8\pi^4}\zeta\left(\frac{9}{2}\right) - \frac{75}{8\pi^2}\zeta\left(\frac{5}{2}\right).$$

Let $m = 2$ and $s = 11, 13$ in (3.10). Then

$$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right)^{-2}}{\left(\sqrt{k} + \sqrt{k+1}\right)^5} = \frac{10395}{32\pi^5}\zeta\left(\frac{11}{2}\right) - \frac{945}{32\pi^3}\zeta\left(\frac{7}{2}\right).$$
$$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right)^{-2}}{\left(\sqrt{k} + \sqrt{k+1}\right)^7} = \frac{135135}{32\pi^6}\zeta\left(\frac{13}{2}\right) - \frac{7245}{16\pi^4}\zeta\left(\frac{9}{2}\right) + \frac{27}{8\pi^2}\zeta\left(\frac{5}{2}\right).$$

Example 4.2. Let $m = 1$ and $s = 7, 9$ in (3.12). Then

$$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{\sqrt{2k+1}} + \frac{1}{\sqrt{2k+3}}\right)^{-1}}{\left(\sqrt{2k+1} + \sqrt{2k+3}\right)^4} = \frac{1}{2} + 105\frac{4\sqrt{2}-1}{32\sqrt{2}\pi^3}\zeta\left(\frac{7}{2}\right) - \frac{\sqrt{2}-1}{2\sqrt{2}\pi}\zeta\left(\frac{3}{2}\right)$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{\sqrt{2k+1}} + \frac{1}{\sqrt{2k+3}}\right)^{-1}}{\left(\sqrt{2k+1} + \sqrt{2k+3}\right)^6} = \frac{1}{2} + 945\frac{8\sqrt{2}-1}{64\sqrt{2}\pi^4}\zeta\left(\frac{9}{2}\right) - 75\frac{2\sqrt{2}-1}{16\sqrt{2}\pi^2}\zeta\left(\frac{5}{2}\right).$$

Ramanujan shown that the higher powers of $S_{\frac{5}{2}}(n)$ are not so neat as (1.2) and (1.3). In the following example, we express $S_{\frac{5}{2}}(n)$ and $S_{\frac{7}{2}}(n)$ in a single infinite series.

Example 4.3. Let $s = 7$ and $m = 1$ in (3.14). Then

$$1^{5/2} + 2^{5/2} + \ldots + n^{5/2} = \frac{35}{64\pi^3}\zeta\left(\frac{7}{2}\right) + \frac{2}{7}n^{\frac{7}{2}} + \frac{1}{2}n^{\frac{5}{2}} + \frac{41}{168}n^{\frac{3}{2}} - \frac{1}{28}n^{\frac{1}{2}} + \frac{1}{168}\sum_{k=0}^{\infty} \frac{2n+2k+1+5\sqrt{(n+k)(n+k+1)}}{\left(\sqrt{n+k} + \sqrt{n+k+1}\right)^5}.$$

Let $s = 9$ and $m = 1$ in (3.14). Then

$$1^{7/2} + 2^{7/2} + \ldots + n^{7/2} = \frac{975}{2304\pi^4}\zeta\left(\frac{9}{2}\right) + \frac{11}{9}n^{\frac{9}{2}} + \frac{137}{288}n^{\frac{7}{2}} + \frac{7}{24}n^{\frac{5}{2}} - \frac{29}{2304}n^{\frac{1}{2}} - \frac{1}{2304}\sum_{k=0}^{\infty} \frac{10n+10k+5+14\sqrt{(n+k)(n+k+1)}}{\left(\sqrt{n+k} + \sqrt{n+k+1}\right)^7}.$$

4.2 Summation of other infinite series

The following infinite series

$$\sum_{k=0}^{\infty} \frac{(x+k+1)^{m/2}}{\left(\sqrt{x+k} + \sqrt{x+k+1}\right)^{s+1}} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(x+k)^{m/2}}{\left(\sqrt{x+k} + \sqrt{x+k+1}\right)^{s+1}}.$$

can be evaluated through the expression

$$\sum_{n=0}^{m} A_{m-n}\Phi(m, n, s-2n).$$

Where A's are constant and Φ is still defined by (2.1). The following are evaluation of these type of infinite series through Φ.

Example 4.4. Consider the following expression

$$\sum_{k=0}^{\infty} \frac{2x+2k+1+A\sqrt{(x+k)(x+k+1)}}{\left(\sqrt{x+k} + \sqrt{x+k+1}\right)^s} = \Phi(0, x, s-2) + (A-2)\Phi(1, x, s). \tag{4.1}$$

Let $A=1$ in (4.1). Then

$$\Phi(0,x,s-2)-3\Phi(1,x,s)=\sum_{k=0}^{\infty}\frac{\sqrt{x+k}^{3}+\sqrt{x+k+1}^{3}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}}. \tag{4.2}$$

Let $A=-3$ in (4.1) and replace s by $s+2$. Then

$$\Phi(0,x,s)-\Phi(1,x,s+2)=\sum_{k=0}^{\infty}\frac{\sqrt{x+k+1}^{3}-\sqrt{x+k}^{3}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}}. \tag{4.3}$$

Adding and subtracting (4.2) and (4.3), then

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\sqrt{x+k+1}^{3}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}} \\ &=\frac{1}{2}\left[\Phi(0,x,s-2)-3\Phi(1,x,s)+\Phi(0,x,s)-\Phi(1,x,s+2)\right].\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\sqrt{x+k}^{3}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}}= \\ &=\frac{1}{2}\left[\Phi(0,x,s-2)-3\Phi(1,x,s)-\Phi(0,x,s)+\Phi(1,x,s+2)\right].\end{aligned} \tag{4.5}$$

Example 4.5. Consider the following expression

$$\begin{aligned}\sum_{k=0}^{\infty}\Bigg[&\frac{(x+k)^2+(x+k+1)^2}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s}}+A\frac{\sqrt{(x+k)(x+k+1)}(2x+2k+1)}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s}} \\ &+B\frac{(x+k)(x+k+1)}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s}}\Bigg]=\Phi(0,x,s-4)+(A-4)\Phi(1,x,s-2) \\ &\qquad +(B-2A+2)\Phi(2,x,s).\end{aligned}$$

Let $A=1$ and $B=1$ in (4.1). Then

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(x+k)^{5/2}-(x+k+1)^{5/2}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}} \\ &\quad=\Phi(0,x,s-4)-3\Phi(1,x,s-2)+\Phi(2,x,s).\end{aligned} \tag{4.6}$$

Let $A=-1$ and $B=1$ in (4.1). Then

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(x+k+1)^{5/2}+(x+k)^{5/2}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}} \\ &\quad=\Phi(0,x,s-2)-5\Phi(1,x,s)+6\Phi(2,x,s+2).\end{aligned} \tag{4.7}$$

Adding and subtracting (4.6) and (4.7), then

$$\sum_{k=0}^{\infty} \frac{(x+k+1)^{5/2}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}} = \frac{1}{2}\left[\Phi(0,x,s-4) - 3\Phi(1,x,s-2)\right.$$
$$\left. +\Phi(2,x,s) + \Phi(0,x,s-4) - 5\Phi(1,x,s-2) + 6\Phi(2,x,s)\right].$$
$$\sum_{k=0}^{\infty} \frac{(x+k)^{5/2}}{\left(\sqrt{x+k}+\sqrt{x+k+1}\right)^{s+1}} = \frac{1}{2}\left[\Phi(0,x,s-4) - 3\Phi(1,x,s-2)\right.$$
$$\left. +\Phi(2,x,s) - \Phi(0,x,s-2) + 5\Phi(1,x,s) - 6\Phi(2,x,s+@)\right].$$

5 Conclusion

The summation of certain infinite series $\Phi(m,0,n)$ involving square root of powers of natural numbers has been studied in this paper. The sum of powers of square root of first n natural numbers has been expressed in terms of infinite series defined in (2.1) and Riemann zeta function. Some numerical examples are given to evaluate summation of the infinite series $\Phi(m,0,n)$ and sum of powers of square root of first n natural numbers.

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