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On rational fractions not expressible as a sum of three unit fractions

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Abstract: Of those fractions (a/b < 1) that can not be expressed as a sum of three unit fractions, many can be written in terms of three unit fractions if the smallest denominator is $\lfloor b/a \rfloor$ and the next largest denominator is < 0. General expressions are given for some specific classes of these. Two examples of Yamamoto are reconsidered.

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Almost all rational fractions (a/b, $0 < a < b$) can be expressed as the sum of three unit fractions [4, 5]

$$\frac{a}{b} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \quad 0 < x \le y \le z, \tag{1}$$

where a, b, x, y and z are integers. It follows from (1) that $b/a < x < 3b/a$ [1], but there are often solutions corresponding to $x = \lfloor b/a \rfloor + 1$ [2].

A list of fractions (a = 8, 9, ... 18 with prime b) that can not be expressed in this way was given by Webb [6]. Here I consider the 134 fractions in that list for which $a < b$. All except 18 of these (Table 1) can be expressed in terms of three unit fractions if two changes are made to (1). The first is that $x = \lfloor b/a \rfloor$, from which $1/x > a/b$, and so y < 0, which is the second modification. Both of these modifications of (1) are present in a series of general expressions, such as

$$\frac{18}{36n+5} = \frac{1}{2n} - \frac{1}{2n(7n+1)} + \frac{1}{2(7n+1)(36n+5)} \tag{2}$$

Table 1. Fractions in Webb's [6] list that can not be expressed in three unit fractions using (1), (2) or the other similar expressions given by Sierpiński [3], (4-6) or (7).

a	b	a	b	a	b	a	b	a	b
8	11	15	19	17	23	18	31	18	193*
12	29	15	23	17	41	18	47	18	281
14	17	15	53	18	23	18	59		
14	19	16	23	18	29	18	137		

* If $x = \lfloor b/a \rfloor - 1$, a solution is $(x, y, z) = (9, -56, 97272)$.

for integral $n > 0$, reported by Sierpiński [3]. Equation (2) provides a solution for 18/41, 18/113, 18/149 and 18/761 (corresponding to $n = 1$, 3, 4 and 21, respectively) each of which appears on Webb's [6] list. Other expressions in this series [3] provide similar expansions for 18/61, 18/131, 18/223 and 18/239, each of which also appears on Webb's [6] list, and there are other solutions if $z < 0$ is also permitted.

These observations prompted the reconsideration of two examples given by Yamamoto [7]. First, for prime $b \geq 11$, $(b-1)/b$ can not be expressed as the sum of three unit fractions (1). This is also the case for some non-prime b, for example $b = 14$, 16, 21, 25, …, although there are instances where (1) does apply for non-prime b (such as $a/b = 8/9$, for which $(x, y, z) = (2, 3, 18)$ is a solution). Yamamoto's [7] second example is that $(b-1)/2b$, for prime $b \geq 29$, can not be expressed as the sum of three unit fractions (1). This is also the case for some non-prime b, for example $b = 25$, 33, 49, 65, …, although there are instances where (1) does apply for non-prime b (such as $a/b = 4/9$, for which $(x, y, z) = (3, 10, 90)$ is a solution). There is a solution for each of these examples if $x = \lfloor b/a \rfloor$ and $y < 0$ in (1), for example

$$\frac{m-1}{m} = 1 - \frac{1}{m-1} + \frac{1}{m(m-1)} \tag{3}$$

for integral $m > 1$, and

$$\frac{m-1}{2m} = \frac{1}{2} - \frac{1}{m} + \frac{1}{2m} \tag{4}$$

for integral $m > 1$, as can be confirmed by simplifying the RHS in each case. Equations (3) and (4) account, respectively, for 4 and 5 fractions in Webb's [6] list ($m = 11$, 13, 17, 19 and $m = 17$, 19, 29, 31, 38, respectively).

Equations (3) and (4) are related by the following lemma.

Lemma 1. If $x = \lfloor b/a \rfloor$ and $y < 0$ in (1), then $a/b = (m-1)/km$, for integral $k > 1$ and $m > 0$, can be expanded as

$$\frac{m-1}{km} = \frac{1}{k} - \frac{1}{(k-1)m} + \frac{1}{k(k-1)m}. \tag{5}$$

Proof. Equation (5) can be confirmed by simplifying the RHS.

Remark. Equation (5), using k ranging from 3 to 1020, accounts for 55 fractions (Table 2) in Webb's [6] list in addition to the 9 mentioned above.

Table 2. Fractions in Webb's [6] list that can be expressed in three unit fractions using (5).

a	b
8	241
10	61
10	181
12	37
12	73
12	97
12	193
12	433
12	577
12	1129
12	1657
12	1873

a	b
12	2521
12	2593
12	3433
12	10369
12	12049
12	12241
13	53
13	79
15	61
15	151
15	271

a	b
15	541
15	1171
15	4201
16	97
16	113
16	193
16	241
16	257
16	421
16	577
16	593

a	b
16	641
16	769
16	1201
16	1489
16	2113
16	2689
16	3169
16	3361
16	4801
16	4993
17	1123

a	b
18	73
18	109
18	181
18	379
18	397
18	541
18	613
18	811
18	1009
18	1297
18	2269

Table 3. Fractions in Webb's [6] list that can be expressed in three unit fractions using (6).

a	b
8	131
10	43
13	67
13	211

a	b
14	59
14	353
15	47
15	79

a	b
15	137
15	197
15	1103
15	1367

a	b
16	421
17	53
17	71

a	b
17	157
18	113
18	821

Another 18 of these fractions (Table 3) can be expressed in a form similar to that of (5).

Lemma 2. If $(ax - b) \mid (x - 1)$, then for $b = (acx + x - 1)/c$

$$\frac{a}{b} = \frac{1}{x} - \frac{1}{bc} + \frac{1}{bcx}. \tag{6}$$

where $c = (x - 1)/(ax - b)$ and $x = \lfloor b/a \rfloor$.

Another three of Webb's [6] fractions (9/11, 15/17, 17/19) can be expressed differently.

Lemma 3. For n =1, 2, ... and m = 1, 2, ...

$$\frac{mn - m + 1}{mn + 1} = 1 - \frac{1}{n} + \frac{1}{n(mn+1)}. \tag{7}$$

Proof. Equation (7) can be confirmed by simplifying the RHS.

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