An explicit estimate
for the Barban and Vehov weights

Djamel Berkane
Department of Mathematics
University of Blida, Algeria
e-mail: djaber72@univ-blida.dz

Abstract: We show that
\[
\sum_{1 \leq n \leq N} \left( \sum_{d|n} \lambda_d \right)^2 \frac{1}{n} \ll \frac{\log N}{\log z},
\]
where \( \lambda_d \) is a real valued arithmetic function called the Barban and Vehov weight and we give
an explicit version of a Theorem of Barban and Vehov which has applications to zero-density
theorems.

Keywords: Explicit estimates, Möbius function, Selberg sieve.

AMS Classification: Primary: 11N37; Secondary: 11N35, 11N05

1 Introduction

While studying an optimisation problem close to the one that classical initial founding of
the Selberg sieve for prime numbers, Barban and Vehov in [1] noticed the property, valid for
\( N \geq z > 1 \)
\[
\sum_{1 \leq n \leq N} \left( \sum_{d|n, d \leq z} \mu(d) \frac{\log(z/d)}{\log z} \right)^2 \ll \frac{N}{\log z}.
\]
The novelty of this estimate is that no additional \( +O(z^2) \) arises, as it does when using a direct
approach. This enables us to avoid the condition \( N \geq z^2 \log(z) \). One of the consequences of this
estimate is the result
\[
\sum_{n \geq 1} \left( \sum_{d|n, d \leq z} \mu(d) \frac{\log(z/d)}{\log z} \right)^2 / n^\omega = \mathcal{O}_c(1),
\]
valid for any constant $c > 0$ and provided that $\omega \geq 1 + c(\log z)^{-1}$. This estimation would be sufficient to be valid for a fixed constant $c > 0$.

The second novelty in (Barban and Vehov, 1968) comes from the fact that they consider the weights

$$\lambda_d = \begin{cases} 
\mu(d) & \text{when } d \leq z, \\
\mu(d) \frac{\log(z^2/d)}{\log z} & \text{when } z \leq d \leq z^2, \\
0 & \text{when } d > z^2.
\end{cases} \quad (1)$$

they consider in fact slightly more general weights with a $y$ instead of the $z^2$ that we use here. They proved that

$$\sum_{1 \leq n \leq N} \left( \sum_{d|n} \lambda_d \right)^2 \ll \frac{N}{\log(y/z)}.$$ 

They sketched a proof and later proofs were given later by Motohashi [8] (see Motohashi [10, section 1.3]) and Graham [5]. The estimate above has been used by Motohashi [9] and Jutila [7] to prove zero-density theorem for $L$-functions which are sensitive near $\sigma = 1$. In this present work, we propose to give via a classical elementary proof, an explicit version of a Theorem of Barban and Vehov jointly with a similar result for the quantity:

$$\sum_{1 \leq n \leq N} \left( \sum_{d|n} \lambda_d \right)^2 / n.$$ 

Our main theorem is the following

**Theorem 1.1.** When $N \geq z > 1$, we have

$$\sum_{1 \leq n \leq N} \left( \sum_{d|n} \lambda_d \right)^2 / n \ll \frac{\log N}{\log z}.$$ 

When $N \leq z$, the sum simply vanishes for every summand does and the constant implied by the $\ll$-symbol is explicitly given.

**Notation**

We denote by $\tau(n)$ the number of (positive) divisors of $n$, and we use here the notation $f = O^*(g)$ to mean that $|f| \leq g$.

**2 Arithmetical lemmas**

Let us begin by giving the following general version in the estimate of the summatory function of the M"obius function with coprimality restrictions

**Lemma 2.1.** For any $x \geq 1$, $\varepsilon \geq 0$ and for any integer $r \geq 1$, we have

$$\left| \sum_{n \leq x, (n,r)=1} \frac{\mu(n)}{n^{\varepsilon+1}} \right| \leq 1.$$
In [6] it is studied the classical case $\varepsilon = 0$.

**Proof.** First, a direct summation by parts gives us

$$
x^{\varepsilon} \sum_{n \leq x} \frac{\mu(n)}{n^{\varepsilon+1}} - \sum_{n \leq x} \frac{\mu(n)}{n} = \varepsilon x^{\varepsilon} \int_1^x \left( \sum_{n \leq t} \frac{\mu(n)}{n} \right) \frac{dt}{t^{\varepsilon+1}}.
$$

(2)

So, the Lemma follows on recalling part of [6, Lemma 10.2]

$$
| \sum_{n \leq x} \mu(n)n^{-1} | \leq 1, \quad (x \geq 1).
$$

Applying the choice $\varepsilon \leq \frac{\log 2}{\log x}$ in (2), let us deduce also the following consequence

$$
|x^{\varepsilon} \sum_{n \leq x} \frac{\mu(n)}{n^{\varepsilon+1}} - \sum_{n \leq x} \frac{\mu(n)}{n}| \leq 1,
$$

uniformly in $x > 1$ and $r$. \hfill \Box

Here and below $e$ represents Napier’s constant and $p$ a prime number.

**Lemma 2.2.** For $x \geq 1$, we have

$$
\sum_{m \leq x} \frac{\mu^2(m)}{\sqrt{m}} \leq 1.33\sqrt{x}.
$$

**Proof.** It suffices to use a summation by parts together with the known result

$$
\sum_{m \leq x} \mu^2(m) - 6x/\pi^2 = O^*(0.1333\sqrt{x}).
$$

\hfill \Box

**Lemma 2.3.** For $p^\varepsilon \leq e^\varepsilon$, we have

$$
\log(1 - \frac{1}{p^{\varepsilon+1}}) - \log(1 - \frac{1}{p}) \leq \varepsilon \frac{\log p}{p} + \frac{17/50}{p^2},
$$

provided that $c \leq \log(1 + 1/p)$.

**Proof.** Setting $t = 1/p^{\varepsilon}$, $x = 1/p$, we just have to prove that

$$
F(x, t) = \log(1 - tx) - \log(1 - x) + x \log t - 17x^2/50,
$$

is non positive function. The first derivatives on $t$, show that $F(x, t)$ is not more than $F(x, 1/(x + 1))$, provided that $\varepsilon \leq \log(1 + x)/\log(1/x)$, which equivalent to $1/(x + 1) \leq t \leq 1$. We conclude the proof after a report that $F(x, 1/(x + 1))$ is non positive if $0 < x \leq 1/2$. \hfill \Box

37
Lemma 2.4. We have for \( x \leq 5 \)

\[
| \sum_{n \leq x} \frac{\mu(n)}{n^{\varepsilon+1}} \log \frac{x}{n} | \leq 2.3 \frac{r}{\phi(r)}.
\]

Proof. Indeed, The best value that the left term can set is \( \sum_{n \leq 5} \frac{\log(5/n)}{n} \) for \((r, 30) = 1 \). This is no more than 2.3. \( \square \)

Lemma 2.5. We have for \( x > 1, x^\varepsilon \leq e^c \)

\[
\sum_{n \leq x} \sum_{m \mid n \atop (m, r) = 1} \frac{\mu(m)}{m^\varepsilon} \tau(n/m) \leq e^{c + 31/200} x^{r/\phi(r)}.
\]

provided that \( c \leq \log(1 + 1/x) \).

Proof. First, let us remark that the left hand side is equal and verify

\[
x \sum_{n \leq x} \sum_{m \mid n \atop (m, r) = 1} \frac{\mu(m)}{m^\varepsilon} \tau(n/m) x \leq x \sum_{n \leq x} \sum_{m \mid n \atop (m, r) = 1} \frac{\mu(m)}{m^\varepsilon} \tau(n/m).
\]

Now, we write the right inequality as

\[
x \prod_{p \leq x, \nu \geq 0 \atop p \nmid r} p^{-\nu} \prod_{p \leq x, \nu \geq 0 \atop p \mid r} \left[ 1 + (1 - p^{-\varepsilon})(\sum_{\nu \geq 0} p^{-\nu}) \right] \leq x \prod_{p \leq x, \nu \geq 0 \atop p \nmid r} 1/(1 - p^{-\varepsilon}) \prod_{p \leq x} 1 - p^{-\varepsilon-1}
\]

\[
\leq x \prod_{p \mid r} \frac{1}{1 - 1/p} \exp(\sum_{p \leq x} S(p)),
\]

where \( S(p) \) is the majored quantity in Lemma 2.3. Hence, the Lemma readily follows on taking \( \varepsilon \leq c(\log x)^{-1} \) and recalling that

\[
\sum_p \frac{1}{p^2} \leq 0.452247421 \text{ and } \sum_{p \leq x} \frac{\log x}{x} \leq \log(x).
\]

On letting \( c \) go to 0, we obtain for the classical case \( \varepsilon = 0 \)

\[
\frac{1}{x} \sum_{n \leq x} \sum_{m \mid n \atop (m, r) = 1} \mu(m) \tau(n/m) \leq 1.168 \frac{r}{\phi(r)}.
\]

\( \square \)

At this level, we quote the following result

Lemma 2.6. Let \( x > 1 \) be a fixed real parameter and \( x^\varepsilon \leq e^{1/5} \). We have

\[
| \sum_{n \leq x} \frac{\mu(n)}{n^{\varepsilon+1}} \log \frac{x}{n} | \leq 2.86 \frac{r}{\phi(r)}.
\]
Furthermore, there exists an infinity of squarefree number $r$ such that

$$\max_{x \geq 1} \left| \sum_{\substack{n \leq x \quad (r,n) = 1 \quad \sigma \in +1}} \frac{\mu(n)}{n+1} \log \frac{x}{n} \right| \geq e^{-\gamma} \frac{r}{\phi(r)} + o(1).$$

**Proof.** At first, according to the previous Lemma the choice $c = 1/5$ is valid when $x > 5$. So, recalling the explicit upper bound of the average value of divisor function given in [3]

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1) x + O^*(0.961x^{1/2}), \quad (x \geq 1),$$

with $\gamma$ the Euler’s constant, let us write for $x > 5$

$$\sum_{n \leq x} \sum_{\substack{m/n \quad (m,r) = 1 \quad \sigma \in +1}} \frac{\mu(m)}{m^{\varepsilon}} \tau(n/m) = \sum_{\substack{m \leq x \quad (m,r) = 1 \quad \sigma \in +1}} \frac{\mu(m)}{m^{\varepsilon+1}} \sum_{n \leq x/m} \tau(n)$$

$$= x \sum_{\substack{m \leq x \quad (m,r) = 1 \quad \sigma \in +1}} \frac{\mu(m)}{m^{\varepsilon+1}} \log \frac{x}{m} + (2\gamma - 1) x \sum_{\substack{m \leq x \quad (m,r) = 1 \quad \sigma \in +1}} \frac{\mu(m)}{m^{\varepsilon+1}}$$

$$+ O^*(0.961 \sqrt{x} \sum_{\substack{m \leq x \quad (m,r) = 1 \quad \sigma \in +1}} \frac{\mu^2(m)}{m^{\varepsilon+1/2}}).$$

Thus, the upper bound of the previous Lemma and Lemmas 2.1, 2.2, we reach

$$\left| \sum_{\substack{m \leq x \quad (m,r) = 1 \quad \sigma \in +1}} \frac{\mu(m)}{m^{\varepsilon+1}} \log \frac{x}{m} \right| \leq e^{c+31/200} \frac{r}{\phi(r)} + 1.433,$$

for $\varepsilon \leq c(\log x)^{-1}$. The Lemma 2.4 takes care of the small values of $x$, so we deduce easily.

For the second part, indeed, for any $L \geq 1$ corresponds the squarefree number $r = \prod_{p \leq L} p$, which verify $\theta(L) = \log r$. Then

$$\max_{x \geq 1} \left| \sum_{\substack{n \leq x \quad (r,n) = 1}} \frac{\mu(n)}{n^{\varepsilon+1}} \log \frac{x}{n} \right| \geq \log \log r + o(1).$$

But

$$\frac{r}{\phi(r)} = \prod_{p \leq L} \left(1 - 1/p\right)^{-1} \sim e^\gamma \log L \sim e^\gamma \log \log r,$$

we conclude the proof. \[\Box\]

**Lemma 2.7.** Let $x \geq 1$ be a fixed real parameter. We have

$$\sum_{n \leq x} \frac{\mu^2(n)}{\phi(n)} \leq \log x + 1.4709.$$
Proof. It is a result given by [11] for the case $d = 1$ on the estimation of

$$\sum_{n \leq x, (n,d)=1} \frac{\mu^2(n)}{\phi(n)}.$$ 

\[ \square \]

**Lemma 2.8.** For $\omega \geq 1$ and $r$ squarefree number, we have

$$\frac{1}{r^{2(\omega-1)}} \prod_{p|r}(p^{\omega} - 1) < \frac{\phi(r)}{r}$$

Proof. Since our product can become

$$r \prod_{p|r} \frac{p^{\omega} - 1}{p^{2\omega}},$$

and by the fact that $\frac{p^{\omega} - 1}{p^{2\omega}}$ is a decreasing function, which takes its maximum at $\omega = 1$, we deduce easily. \[ \square \]

### 3 Some results connected to weights $\lambda_d$

We shall need explicit estimates connected to the weight (1).

Let us put

$$\mathcal{L}(y, d) = \begin{cases} \mu(d) \log \frac{y}{d} & \text{if } d \leq y, \\ 0 & \text{if not.} \end{cases}$$

We notice that the following decomposition takes place

$$\lambda_d = (\mathcal{L}(z^2, d) - \mathcal{L}(z, d)) / \log z,$$

who allows to deduct estimations concerning the $\lambda_d$ of those concerning $\mathcal{L}(y, d)$.

**Lemma 3.1.** When $0 \leq \omega - 1 \leq (5 \log y)^{-1}$, the quantity

$$\mathcal{R}(r, y, \omega) = \sum_{d \geq 1} \frac{\mathcal{L}(y, rd)}{(rd)^{\omega}},$$

verify

$$|\mathcal{R}(r, y, \omega)| \leq 2.86 \frac{1}{r^\omega} \frac{r}{\phi(r)}.$$

Proof. Indeed, the Lemma 2.6 and the writing

$$\mathcal{R}(r, y, \omega) = \frac{\mu(r)}{r^\omega} \mathcal{R}(r, \frac{y}{r}).$$
where
\[
R(r, x) = \sum_{d \leq x, (d, r) = 1} \frac{\mu(d) \log \frac{x}{d}}{d^{\omega}},
\]
give us
\[
|R(r, x)| \leq 2.86 \frac{r}{\phi(r)},
\]
so, the Lemma follows readily.

Let \([a, b]\) the least common multiple of \(a\) and \(b\). We note the following:

**Lemma 3.2.** For any \(y > 1\), we have
\[
\sum_{d_1, d_2 \geq 1} \frac{L(y, d_1) L(y, d_2)}{[d_1, d_2]^{\omega}} \leq 8.18 (\log y + 1.4709),
\]
as soon as \(0 < \omega - 1 \leq (5 \log y)^{-1}\).

**Proof.** Let us denote by \(S(y, \omega)\) the quantity to evaluate. First, we use Selberg diagonalization process. We start by writing
\[
S(y, \omega) = \sum_{d_1, d_2} \frac{L(y, d_1) L(y, d_2) (d_1, d_2)^{\omega}}{d_1^{\omega} d_2^{\omega}} = \sum_{d_1, d_2} \mu^2(d_1) \mu^2(d_2) L(y, d_1) L(y, d_2) (d_1, d_2)^{\omega} \frac{\omega}{d_1^{\omega} d_2^{\omega}}.
\]

Now, let us define the function \(\Phi_{\omega}(r) = \prod_{p|r} (p^{\omega} - 1)\), so that for \(r\) squarefree number, we obtain \(r^{\omega} = (\Phi_{\omega} \ast 1)(r)\). From this we infer that
\[
S(y, \omega) = \sum_{r \leq y} \mu^2(r) \Phi_{\omega}(r) \left( \sum_{d \geq 1} \frac{L(y/r, rd)^{\omega}}{(rd)^{\omega}} \right)^2 = \sum_{r \leq y} \mu^2(r) \Phi_{\omega}(r) \mathcal{R}(r, y/r, \omega)^2.
\]

Thus, by conjugating the Lemmas 2.7, 2.8 and 3.1, the Lemma follows.

**Lemma 3.3.** For \(x \geq y > 1\), we have
\[
\sum_{n \leq x} \left( \sum_{d \mid n} L(y, d) \right)^2 / n \leq 10 (\log y + 1.4709) (5 \log x + 1).
\]

**Proof.** First, according to the Rankin’s method [12, Lemma 2], we can write for any \(\varepsilon > 0\)
\[
\sum_{n \leq x} \left( \sum_{d \mid n} L(y, d) \right)^2 / n \leq \sum_{n \leq x} \left( \frac{\sum_{d \mid n} L(y, d)^2}{n} \frac{x}{n} \right)^\varepsilon \leq x^\varepsilon \sum_{n \geq 1} \left( \sum_{d \mid n} L(y, d) \right)^2 / n^{1+\varepsilon}.
\]
Now, we choose $\varepsilon = (5 \log x)^{-1}$ and define $\omega = 1 + \varepsilon$. We expand the square and find that
\[
\sum_{n \leq x} \left( \sum_{d | n} \mathcal{L}(y, d) \right)^2 / n \leq x^\varepsilon \sum_{d_1, d_2} \mu^2(d_1) \mu^2(d_2) \mathcal{L}(y, d_1) \mathcal{L}(y, d_2) \mathcal{L}(y, d) \zeta(1 + \varepsilon)
\leq \mathcal{S}(y, \omega) \zeta(\omega) x^\varepsilon,
\]
where $\mathcal{S}(y, \omega)$ is the quantity we have evaluated in the precedent Lemma. Finally, by observing that $x^\varepsilon = e^{1/5}$ and taking
\[
\zeta(\omega) \leq \frac{\omega}{\omega - 1},
\]
since $\omega$ is real and close to 1 (See[2, Corollary 1] as well as the in [4, Lemma 2.3]), we conclude the proof.

4 Proof of main Theorem

We start with the decomposition (3) of $\lambda_d$:
\[
\lambda_d = \mu(d)^2 \log(z^2/d) \log z \mathbb{1}_{d \leq z^2} - \mu(d) \log(z/d) \log z \mathbb{1}_{d \leq z}.
\]
Since $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, this leads us to
\[
(\log z)^2 \sum_{1 \leq n \leq N} \left( \sum_{d | n} \lambda_d \right)^2 / n \leq \sum_{1 \leq n \leq N} \left( \sum_{d | n} \mu(d) \log(z^2/d) \log z \mathbb{1}_{d \leq z^2} \right)^2 / n + 2 \sum_{1 \leq n \leq N} \left( \sum_{d | n} \mu(d) \log(z/d) \log z \mathbb{1}_{d \leq z} \right)^2 / n.
\]
Finally, by applying the precedent Lemma for each summand, we get then
\[
\sum_{1 \leq n \leq N} \left( \sum_{d | n} \lambda_d \right)^2 / n \leq 60 \log z + 1 \log N (5 \log N + 1),
\]
when $N \geq z > 1$. The Theorem follows.

5 An explicit Theorem of Barban and Vehov

We can also explicitly obtain the following Lemma:

Lemma 5.1. For $x > 1$ and $\omega \geq 1 + c(\log x)^{-1}$, we have
\[
\sum_{n \geq 1} \left( \sum_{d | n} \mathcal{L}(x, d) \right)^2 / n^\omega \ll_c (\log x)^2,
\]
provided that $c \leq \log(1 + 1/x)$.

Proof. We just have to treat the case $\omega = 1 + c(\log x)^{-1}$. Following the notations and the same first steps in proof of Lemma 3.3, we find that
\[
\sum_{n \geq 1} \left( \sum_{d | n} \mathcal{L}(x, d) \right)^2 / n^\omega \leq \mathcal{S}(x, \omega) \zeta(\omega).
\]
Thus, applying Lemma 3.2 and using $\zeta(\omega) \leq \frac{\omega}{\omega - 1}$, give the result. 

42
This Lemma yields readily the next one

**Theorem 5.1** (Barban and Vehov). *For* \( x > 1 \), *we have*

\[
\sum_{n \geq 1} \left( \sum_{d \mid n} \lambda_d \right)^2 / n^\omega \ll_{c} 1,
\]

*as soon as* \( \omega \geq 1 + c \log x \) *and* \( c \leq \log(1 + 1/x) \).

*Proof.* It is enough to use the decomposition (3) and the precedent Lemma on each summand which appears. \( \square \)

**References**


