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An explicit estimate for the Barban and Vehov weights

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Abstract: We show that

$$\sum_{1\leq n\leq N}\Big(\sum_{d|n}\lambda_d\Big)^2/n \ll \frac{\log N}{\log z},$$

where λ_d is a real valued arithmetic function called the Barban and Vehov weight and we give an explicit version of a Theorem of Barban and Vehov which has applications to zero-density theorems.

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1 Introduction

While studying an optimisation problem close to the one that classical initial founding of the Selberg sieve for prime numbers, Barban and Vehov in [1] noticed the property, valid for $N \geq z > 1$

$$\sum_{1\leq n\leq N}\Big(\sum_{\substack{d|n,\\ d\leq z}}\mu(d)\frac{\log(z/d)}{\log z}\Big)^2 \ll N/\log z.$$

The novelty of this estimate is that no additionnal $+\mathcal{O}(z^2)$ arises, as it does when using a direct approach. This enables us to avoid the condition $N \geq z^2\log(z)$. One of the consequences of this estimate is the result

$$\sum_{n\geq 1}\Big(\sum_{\substack{d|n,\\ d\leq z}}\mu(d)\frac{\log(z/d)}{\log z}\Big)^2/n^{\omega} = \mathcal{O}_c(1),$$

valid for any constant $c > 0$ and provided that $\omega \geq 1 + c(\log z)^{-1}$. This estimation would be sufficient to be valid for a fixed constant $c > 0$.

The second novelty in (Barban and Vehov, 1968) comes from the fact that they consider the weights

$$\lambda_d = \begin{cases} \mu(d) & \text{when } d \leq z, \\ \mu(d)\dfrac{\log(z^2/d)}{\log z} & \text{when } z \leq d \leq z^2, \\ 0 & \text{when } d > z^2. \end{cases} \tag{1}$$

they consider in fact slightly more general weights with a y instead of the z^2 that we use here. They proved that

$$\sum_{1 \leq n \leq N} \Big(\sum_{d|n} \lambda_d \Big)^2 \ll \frac{N}{\log(y/z)}.$$

They sketched a proof and later proofs were given later by Motohashi [8] (see Motohashi [10, section 1.3]) and Graham [5]. The estimate above has been used by Motohashi [9] and Jutila [7] to prove zero-density theorem for L-functions which are sensitive near $\sigma = 1$. In this present work, we propose to give via a classical elementary proof, an explicit version of a Theorem of Barban and Vehov jointly with a similar result for the quantity:

$$\sum_{1 \leq n \leq N} \Big(\sum_{d|n} \lambda_d \Big)^2 / n.$$

Our main theorem is the following

Theorem 1.1. *When $N \geq z > 1$, we have*

$$\sum_{1 \leq n \leq N} \Big(\sum_{d|n} \lambda_d \Big)^2 / n \ll \frac{\log N}{\log z}.$$

When $N \leq z$, the sum simply vanishes for every summand does and the constant implied by the $\ll$-symbol is explicitly given.

Notation

We denote by $\tau(n)$ the number of (positive) divisors of n, and we use here the notation $f = \mathcal{O}^*(g)$ to mean that $| f | \leq g$.

2 Arithmetical lemmas

Let us begin by giving the following general version in the estimate of the summatory function of the Möbius function with coprimality restrictions

Lemma 2.1. *For any $x \geq 1$, $\varepsilon \geq 0$ and for any integer $r \geq 1$, we have*

$$| \sum_{\substack{n \leq x \\ (n,r)=1}} \frac{\mu(n)}{n^{\varepsilon+1}} | \leq 1.$$

In [6] it is studied the classical case $\varepsilon = 0$.

Proof. First, a direct summation by parts gives us

$$x^{\varepsilon} \sum_{\substack{n\le x\\ (n,r)=1}} \frac{\mu(n)}{n^{\varepsilon+1}} - \sum_{\substack{n\le x\\ (n,r)=1}} \frac{\mu(n)}{n} = \varepsilon x^{\varepsilon} \int_1^x \Big(\sum_{\substack{n\le t\\ (n,r)=1}} \frac{\mu(n)}{n}\Big) \frac{dt}{t^{\varepsilon+1}}. \tag{2}$$

So, the Lemma follows on recalling part of [6, Lemma 10.2]

$$|\sum_{\substack{n\le x\\ (n,r)=1}} \mu(n)n^{-1}| \le 1, \ \ (x \ge 1).$$

Applying the choice $\varepsilon \le \dfrac{\log 2}{\log x}$ in (2), let us deduce also the following consequence

$$|x^{\varepsilon} \sum_{\substack{n\le x\\ (n,r)=1}} \frac{\mu(n)}{n^{\varepsilon+1}} - \sum_{\substack{n\le x\\ (n,r)=1}} \frac{\mu(n)}{n}| \le 1,$$

uniformly in $x > 1$ and r. □

Here and below e represents Napier's constant and p a prime number.

Lemma 2.2. *For $x \ge 1$, we have*

$$\sum_{m\le x} \mu^2(m)/\sqrt{m} \le 1.33\sqrt{x}.$$

Proof. It suffices to use a summation by parts together with the known result

$$\sum_{m\le x} \mu^2(m) - 6x/\pi^2 = \mathcal{O}^*(0.1333\sqrt{x}).$$

□

Lemma 2.3. *For $p^{\varepsilon} \le e^{c}$, we have*

$$\log(1 - \frac{1}{p^{\varepsilon+1}}) - \log(1 - \frac{1}{p}) \le \varepsilon \frac{\log p}{p} + \frac{17/50}{p^2},$$

provided that $c \le \log(1 + 1/p)$.

Proof. Setting $t = 1/p^{\varepsilon}$, $x = 1/p$, we just have to prove that

$$F(x,t) = \log(1 - tx) - \log(1 - x) + x\log t - 17x^2/50,$$

is non positive function. The first derivatives on t, show that $F(x,t)$ is not more than $F(x, 1/(x+1))$, provided that $\varepsilon \le \log(1+x)/\log(1/x)$, which equivalent to $1/(x+1) \le t \le 1$. We conclude the proof after a report that $F(x, 1/(x+1))$ is non positive if $0 < x \le 1/2$. □

Lemma 2.4. *We have for* $x \leq 5$

$$|\sum_{\substack{n\leq x\\(r,n)=1}} \frac{\mu(n)}{n^{\varepsilon+1}} \log \frac{x}{n}| \leq 2.3\frac{r}{\phi(r)}.$$

Proof. Indeed, The best value that the left term can set is $\sum_{n\leq 5} \frac{\log(5/n)}{n}$ for $(r,30)=1$. This is no more than 2.3. □

Lemma 2.5. *We have for* $x > 1$, $x^\varepsilon \leq e^c$

$$\sum_{n\leq x} \sum_{\substack{m|n\\(m,r)=1}} \frac{\mu(m)}{m^\varepsilon}\tau(n/m) \leq e^{c+31/200}x\frac{r}{\phi(r)}.$$

provided that $c \leq \log(1+1/x)$.

Proof. First, let us remarque that the left hand side is equal and verify

$$x\sum_{n\leq x} \sum_{\substack{m|n\\(m,r)=1}} \frac{\mu(m)}{m^\varepsilon}\frac{\tau(n/m)}{x} \leq x\sum_{n\leq x} \sum_{\substack{m|n\\(m,r)=1}} \frac{\mu(m)}{m^\varepsilon}\frac{\tau(n/m)}{n}.$$

Now, we write the right inequality as

$$\begin{aligned} x\prod_{\substack{p\leq x,\\p|r}}\sum_{\nu\geq 0} p^{-\nu}\prod_{\substack{p\leq x,\\p\nmid r}}[1+(1-p^{-\varepsilon})(\sum_{\nu\geq 0} p^{-\nu})] &\leq x\prod_{\substack{p\leq x,\\p|r}} 1/(1-p^{-\varepsilon-1})\prod_{p\leq x}\frac{1-p^{-\varepsilon-1}}{1-1/p} \\ &\leq x\prod_{p|r}\frac{1}{1-1/p}\exp(\sum_{p\leq x} S(p)), \end{aligned}$$

where $S(p)$ is the majored quantity in Lemma 2.3. Hence, the Lemma readily follows on taking $\varepsilon \leq c(\log x)^{-1}$ and recalling that

$$\sum_p \frac{1}{p^2} \leq 0.452247421 \text{ and } \sum_{p\leq x}\frac{\log x}{x} \leq \log(x).$$

On letting c go to 0, we obtain for the classical case $\varepsilon = 0$

$$\frac{1}{x}\sum_{n\leq x} \sum_{\substack{m|n\\(m,r)=1}} \mu(m)\tau(n/m) \leq 1.168\frac{r}{\phi(r)}.$$

□

At this level, we quote the following result

Lemma 2.6. *Let* $x > 1$ *be a fixed real parameter and* $x^\varepsilon \leq e^{1/5}$. *We have*

$$|\sum_{\substack{n\leq x\\(r,n)=1}} \frac{\mu(n)}{n^{\varepsilon+1}} \log \frac{x}{n}| \leq 2.86\frac{r}{\phi(r)}.$$

Furthermore, there exists an infinity of squarefree number r such that

$$\underset{x\geq 1}{Max}|\sum_{\substack{n\leq x\\(r,n)=1}}\frac{\mu(n)}{n^{\varepsilon+1}}\log\frac{x}{n}|\geq e^{-\gamma}\frac{r}{\phi(r)}+o(1).$$

Proof. At first, according to the previous Lemma the choice $c=1/5$ is valid when $x>5$. So, recalling the explicit upper bound of the average value of divisor function given in [3]

$$\sum_{n\leq x}\tau(n)=x\log x+(2\gamma-1)\,x+\mathcal{O}^*(0.961x^{\frac{1}{2}}),\ \ (x\geq 1),$$

with γ the Euler's constant, let us write for $x>5$

$$\begin{aligned}\sum_{n\leq x}\sum_{\substack{m|n\\(m,r)=1}}\frac{\mu(m)}{m^{\varepsilon}}\tau(n/m)&=\sum_{\substack{m\leq x\\(m,r)=1}}\frac{\mu(m)}{m^{\varepsilon+1}}\sum_{n\leq x/m}\tau(n)\\&=x\sum_{\substack{m\leq x\\(m,r)=1}}\frac{\mu(m)}{m^{\varepsilon+1}}\log\frac{x}{m}+(2\gamma-1)x\sum_{\substack{m\leq x\\(m,r)=1}}\frac{\mu(m)}{m^{\varepsilon+1}}\\&\qquad+\mathcal{O}^*(0.961\sqrt{x}\sum_{\substack{m\leq x\\(m,r)=1}}\frac{\mu^2(m)}{m^{\varepsilon+\frac{1}{2}}}).\end{aligned}$$

Thus, the upper bound of the previous Lemma and Lemmas 2.1, 2.2, we reach

$$|\sum_{\substack{m\leq x\\(m,r)=1}}\frac{\mu(m)}{m^{\varepsilon+1}}\log\frac{x}{m}|\leq e^{c+31/200}\frac{r}{\phi(r)}+1.433,$$

for $\varepsilon\leq c(\log x)^{-1}$. The Lemma 2.4 takes care of the small values of x, so we deduce easily.

For the second part, indeed, for any $L\geq 1$ corresponds the squarefree number $r=\prod_{p\leq L}p$, which verify $\theta(L)=\log r$. Then

$$\underset{x\geq 1}{Max}|\sum_{\substack{n\leq x\\(r,n)=1}}\frac{\mu(n)}{n^{\varepsilon+1}}\log\frac{x}{n}|\geq\log\log r+o(1).$$

But

$$\frac{r}{\phi(r)}=\prod_{p\leq L}\Big(1-1/p)^{-1}\sim e^{\gamma}\log L\sim e^{\gamma}\log\log r,$$

we conclude the proof. □

Lemma 2.7. *Let $x\geq 1$ be a fixed real parameter. We have*

$$\sum_{n\leq x}\frac{\mu^2(n)}{\phi(n)}\leq\log x+1.4709.$$

Proof. It is a result given by [11] for the case $d = 1$ on the estimation of

$$\sum_{\substack{n\le x,\\(n,d)=1}} \frac{\mu^2(n)}{\phi(n)}.$$

□

Lemma 2.8. *For $\omega \geq 1$ and r squarefree number, we have*

$$\frac{1}{r^{2(\omega-1)}}\prod_{p|r}(p^{\omega}-1) < \frac{\phi(r)}{r}$$

Proof. Since our product can become

$$r\prod_{p|r}\frac{p^{\omega}-1}{p^{2\omega}},$$

and by the fact that $\dfrac{p^{\omega}-1}{p^{2\omega}}$ is a decreasing function, which takes its maximum at $\omega = 1$, we deduce easily. □

3 Some results connected to weights λ_d

We shall need explicit estimates connected to the weight (1).
Let us put

$$\mathcal{L}(y,d) = \begin{cases} \mu(d)\log\frac{y}{d} & \text{if } d \leq y,\\ 0 & \text{if not.}\end{cases}$$

We notice that the following decomposition takes place

$$\lambda_d = \big(\mathcal{L}(z^2,d) - \mathcal{L}(z,d)\big)/\log z, \tag{3}$$

who allows to deduct estimations concerning the λ_d of those concerning $\mathcal{L}(y,d)$.

Lemma 3.1. *When $0 \leq \omega - 1 \leq (5\log y)^{-1}$, the quantity*

$$\mathscr{R}(r,y,\omega) = \sum_{d\geq 1}\frac{\mathcal{L}(y,rd)}{(rd)^{\omega}},$$

verify

$$|\mathscr{R}(r,y,\omega)| \leq 2.86\frac{1}{r^{\omega}}\frac{r}{\phi(r)}.$$

Proof. Indeed, the Lemma 2.6 and the writing

$$\mathscr{R}(r,y,\omega) = \frac{\mu(r)}{r^{\omega}}\mathcal{R}(r,\frac{y}{r}).$$

where

$$\mathcal{R}(r,x) = \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)\log\frac{x}{d}}{d^\omega},$$

give us

$$|\mathcal{R}(r,x)| \le 2.86\,\frac{r}{\phi(r)},$$

so, the Lemma follows readily. □

Let $\big[a,b\big]$ the least common multiple of a and b. We note the following:

Lemma 3.2. *For any $y>1$, we have*

$$\sum_{d_1,d_2\ge 1} \frac{\mathcal{L}(y,d_1)\mathcal{L}(y,d_2)}{\big[d_1,d_2\big]^\omega} \le 8.18(\log y + 1.4709),$$

as soon as $0<\omega-1\le(5\log y)^{-1}$.

Proof. Let us denote by $\mathscr{S}(y,\omega)$ the quantity to evaluate. First, we use Selberg diagonalization process. We start by writing

$$\begin{aligned}\mathscr{S}(y,\omega) &= \sum_{d_1,d_2} \frac{\mathcal{L}(y,d_1)\mathcal{L}(y,d_2)(d_1,d_2)^\omega}{d_1^\omega d_2^\omega}\\ &= \sum_{d_1,d_2} \frac{\mu^2(d_1)\mu^2(d_2)\mathcal{L}(y,d_1)\mathcal{L}(y,d_2)(d_1,d_2)^\omega}{d_1^\omega d_2^\omega}.\end{aligned}$$

Now, let us define the function $\mathit{\Phi}_\omega(r)=\prod_{p|r}(p^\omega-1)$, so that for r squarefree number, we obtain $r^\omega=(\mathit{\Phi}_\omega\star 1\!\!1)(r)$. From this we infer that

$$\begin{aligned}\mathscr{S}(y,\omega) &= \sum_{r\le y}\mu^2(r)\mathit{\Phi}_\omega(r)\Big(\sum_{d\ge1}\frac{\mathcal{L}(y/r,rd)}{(rd)^\omega}\Big)^2\\ &= \sum_{r\le y}\mu^2(r)\mathit{\Phi}_\omega(r)\mathscr{R}(r,y/r,\omega)^2.\end{aligned}$$

Thus, by conjugating the Lemmas 2.7, 2.8 and 3.1, the Lemma follows. □

Lemma 3.3. *For $x\ge y>1$, we have*

$$\sum_{n\le x}\Big(\sum_{d|n}\mathcal{L}(y,d)\Big)^2/n \le 10(\log y+1.4709)(5\log x+1).$$

Proof. First, according to the Rankin's method [12, Lemma 2], we can write for any $\varepsilon>0$

$$\begin{aligned}\sum_{n\le x}\Big(\sum_{d|n}\mathcal{L}(y,d)\Big)^2/n &\le \sum_{n\le x}\frac{\Big(\sum_{d|n}\mathcal{L}(y,d)\Big)^2}{n}\Big(\frac{x}{n}\Big)^\varepsilon\\ &\le x^\varepsilon\sum_{n\ge1}\Big(\sum_{d|n}\mathcal{L}(y,d)\Big)^2/n^{1+\varepsilon}.\end{aligned}$$

Now, we choose $\varepsilon = (5\log x)^{-1}$ and define $\omega = 1+\varepsilon$. We expand the square and find that

$$\begin{aligned}\sum_{n\le x}\Big(\sum_{d|n}\mathcal{L}(y,d)\Big)^2/n &\le x^\varepsilon \sum_{d_1,d_2}\frac{\mu^2(d_1)\mu^2(d_2)\mathcal{L}(y,d_1)\mathcal{L}(y,d_2)}{[d_1,d_2]^{1+\varepsilon}}\zeta(1+\varepsilon)\\ &\le \mathscr{S}(y,\omega)\zeta(\omega)x^\varepsilon,\end{aligned}$$

where $\mathscr{S}(y,\omega)$ it is the quantity that we have evaluated in the precedent Lemma. Finally, by observing that $x^\varepsilon = e^{1/5}$ and taking

$$\zeta(\omega) \le \frac{\omega}{\omega-1},$$

since ω is real and close to 1 (See[2, Corollary 1] as well as the in [4, Lemma 2.3]), we conclude the proof. □

4 Proof of main Theorem

We start with the decomposition (3) of λ_d:

$$\lambda_d = \mu(d)\frac{\log(z^2/d)}{\log z}\mathbb{1}_{d\le z^2} - \mu(d)\frac{\log(z/d)}{\log z}\mathbb{1}_{d\le z}.$$

Since $|a+b|^2 \le 2(|a|^2+|b|^2)$, this leads us to

$$(\log z)^2\sum_{1\le n\le N}\Big(\sum_{d|n}\lambda_d\Big)^2/n \le 2\sum_{1\le n\le N}\Big(\sum_{\substack{d|n,\\ d\le z^2}}\mu(d)\log\frac{z^2}{d}\Big)^2/n + 2\sum_{1\le n\le N}\Big(\sum_{\substack{d|n,\\ d\le z}}\mu(d)\log\frac{z}{d}\Big)^2/n.$$

Finally, by applying the precedent Lemma for each summand, we get then

$$\sum_{1\le n\le N}\Big(\sum_{d|n}\lambda_d\Big)^2/n \le 60\,\frac{\log z+1}{(\log z)^2}(5\log N+1),$$

when $N\ge z>1$. The Theorem follows.

5 An explicit Theorem of Barban and Vehov

We can also explicitly obtain the following Lemma:

Lemma 5.1. *For $x>1$ and $\omega \ge 1+c(\log x)^{-1}$, we have*

$$\sum_{n\ge 1}\Big(\sum_{d|n}\mathcal{L}(x,d)\Big)^2/n^\omega \ll_c (\log x)^2,$$

provided that $c \le \log(1+1/x)$.

Proof. We just have to treat the case $\omega = 1+c(\log x)^{-1}$. Following the notations and the same first steps in proof of Lemma 3.3, we find that

$$\sum_{n\ge 1}\Big(\sum_{d|n}\mathcal{L}(x,d)\Big)^2/n^\omega \le \mathscr{S}(x,\omega)\zeta(\omega).$$

Thus, applying Lemma 3.2 and using $\zeta(\omega)\le \dfrac{\omega}{\omega-1}$, give the result. □

This Lemma yields readily the next one

Theorem 5.1 (Barban and Vehov)**.** *For $x > 1$, we have*

$$\sum_{n\geq 1}\Big(\sum_{d|n}\lambda_d\Big)^2/n^\omega \ll_c 1,$$

as soon as $\omega \geq 1 + c(\log x)^{-1}$ and $c \leq \log(1 + 1/x)$.

Proof. It is enough to use the decomposition (3) and the precedent Lemma on each summand which appears. □

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