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Balancing sequences of matrices with application to algebra of balancing numbers

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Abstract: It is well known that, the problem of finding a sequence of real numbers a_n, $n = 0, 1, 2, \ldots$, which is both geometric ($a_{n+1} = ka_n,\ n = 0, 1, 2, \ldots$) and balancing ($a_{n+1} = 6a_n - a_{n-1},\ a_0 = 0,\ a_1 = 1$) admits an unique solution. In fact, the sequence is $1, \lambda_1, \lambda_1^2, \ldots, \lambda_1^n \ldots$, where $\lambda_1 = 3 + \sqrt{8}$ satisfies the balancing equation $\lambda^2 - 6\lambda + 1 = 0$. In this paper, we pose an equivalent problem for a sequence of real, nonsingular matrices of order two and show that, this problem admits an infinity of solutions, that is there exist infinitely many such sequences.

Keywords: Balancing numbers, Lucas-balancing numbers, Balancing matrix, Lucas-balancing matrix.
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1 Introduction

Behera and Panda [1] recently introduced a number sequence called balancing numbers defined in the following way: A positive integer n is called a balancing number with balancer r, if it is the solution of the Diophantine equation

$$1 + 2 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + r).$$

By slightly modifying this Diophantine equation, Panda and Ray [6] introduced cobalancing numbers as the solutions of the Diophantine equation

$$1 + 2 + \ldots + n = (n + 1) + (n + 2) + \ldots + (n + R),$$

where R is the cobalancer corresponding to the cobalancer n. Balancing numbers B_n and the related Lucas-balancing numbers $C_n = \sqrt{8n^2+1}$ are defined recursively by the formulas $B_{n+1} = 6B_n - B_{n-1}$ with $B_0 = 0,\ B_1 = 1$ and $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1,\ C_1 = 3$ [7, 8]. Both the relations satisfy the balancing equation

$$\lambda^2 - 6\lambda + 1 = 0, \tag{1.1}$$

which admits an unique solution (λ_1, λ_2) with $\lambda_1 = 3+\sqrt{8}$ and $\lambda_2 = 3-\sqrt{8}$. In fact, a sequence of balancing constants,

$$1, \lambda_1, \lambda_1^2, \ldots \lambda_1^n, \ldots, \tag{1.2}$$

satisfies the balancing equation (1.1), where the balancing mean $\lim\limits_{n\to\infty} \dfrac{B_{n+1}}{B_n} = \lambda_1 = 3+\sqrt{8}$. Many properties of balancing numbers and their related sequences are available in the literature. Interested readers can follow [2, 3, 4, 5, 9, 10, 11, 12, 13, 14, 15].

In this paper, we pose an equivalent problem for a sequence of real non-singular matrices of order two and we will show that this problem admits an infinitely many such sequences of balancing constants. Indeed, each sequence is naturally related to each other in terms of the generator of the sequences. Finally, with the help of some basic tools from the theory of matrices to the generators of these balancing sequences, we deduce some of the more familiar balancing and Lucas-balancing identities and the fascinating Binet's formulas for the general terms of balancing and Lucas-balancing numbers.

2 Defining equations

Consider the matrix

$$A = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

and the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the entries x, y, u, v are to be determined subject to the constraint $xv - yu \neq 0$. Notice that the geometric sequence $I, A, A^2, \ldots, A^n, \ldots$ is to be balancing if and only if it will satisfy the balancing equation

$$A^2 = 6A - I, \tag{2.1}$$

that is

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix} = 6\begin{pmatrix} x & y \\ u & v \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

It is clear that for A is not nilpotent (Since we are restricted A to be nonsingular), (2.1) implies to the following identity:

$$A^{n+1} = 6A^n - A^{n-1}, \quad n = 1, 2, 3 \ldots.$$

Further, (2.2) is equivalent to the following system of equations:

$$x^2 - 6x + 1 + yu = 0, \tag{2.3}$$
$$(x + v - 6)y = 0, \tag{2.4}$$
$$(x + v - 6)u = 0, \tag{2.5}$$
$$v^2 - 6v + 1 + yu = 0. \tag{2.6}$$

Now consider the following cases to investigate possible solution sets of the mentioned equations.

First we consider $y = 0$ and observe that (2.3) and (2.6) reduce to balancing equation (1.1) giving the solution sets $x = \{\lambda_1, \lambda_2\}$, where $\lambda_1 = 3 + \sqrt{8},\ \lambda_2 = 3 - \sqrt{8}$. Further if $u = 0$, solution matrices of (2.2) are

$$\Phi_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \Phi_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \Phi_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \Phi_3 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Notice that, the following correlations for the balancing constants $\lambda_1,\ \lambda_2$ are valid for every n.

$$\lambda_1 + \lambda_2 = 6,\ \lambda_1 - \lambda_2 = 2\sqrt{8},\ \lambda_1 \cdot \lambda_2 = 1,\ \lambda_1^2 = 6\lambda_1 - 1,$$
$$\lambda_2^2 = 6\lambda_2 - 1,\ \lambda_1^{n+1} = 6\lambda_1^n - \lambda_1^{n-1},\ \lambda_2^{n+1} = 6\lambda_2^n - \lambda_2^{n-1}.$$

Applying appropriate identities we observe that, each of the sequences $\{\Phi_0^n\}$, $\{\Phi_1^n\}$, $\{\Phi_2^n\}$, $\{\Phi_3^n\}$ is a balancing sequence (1.2). Again for $u \neq 0$, (2.5) reduces to $x+v-6 = 0$ and therefore the solution matrices of (2.2) are given by

$$\Phi_{0u} = \begin{pmatrix} \lambda_1 & 0 \\ u & \lambda 2 \end{pmatrix}, \Phi_{2u} = \begin{pmatrix} \lambda_2 & 0 \\ u & \lambda 1 \end{pmatrix}.$$

It can be easily shown that the general term of the sequence generated by Φ_{0u} will be

$$\Phi_{0u}^n = \begin{pmatrix} \lambda_1^n & 0 \\ B_n u & \lambda_2^n \end{pmatrix},$$

where B_n is the n^{th} balancing number.

Consider the next case when $y \neq 0$. In this case if $u = 0$, (2.3) and (2.6) reduce to the balancing equation (1.1) and (2.5) implies $x + v = 6$. The situation is similar to the previous case when $u \neq 0$. Further when $u \neq 0$, (2.5) reduces to $x = 6 - v$, consistent with (2.4). Substituting the value of x in (2.3) , we get

$$(6 - v)^2 - 6(6 - v) + 1 + yu = 0.$$

Further simplification gives the equation

$$v^2 - 6v + 1 + yu = 0,$$

consistent with (2.6). Thus assuming $y \neq 0,\ u \neq 0$, (2.3) to (2.6) reduce to the following equivalent system:

$$v = 3 \pm \sqrt{8 - yu},\ \ x = 6 - v. \tag{2.7}$$

Indeed, this system of equations will help to investigate various sets of solutions of (2.2) which will be discussed in the following section.

3 Some examples of balancing sequences

Example 3.1. Setting y and u to integer values in (2.7), we observe that, there is a unique pair $y = -1,\ u = 1$ which keeps the radicand positive. In this case we obtain two sets of solutions:

$$x = 0, v = 6, y = -1, u = 1;\ \text{ and }\ \mathrm{x = 6, v = 0, y = -1, u = 1}.$$

The latter set results the balancing matrix $Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$ and the corresponding balancing sequence is

$$I, Q_B, Q_B^2, Q_B^3, \ldots, Q_B^n, \ldots,$$

where $Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}$ [12].

Example 3.2. Now, for the pair $y = u = 0$ in (2.7), results in the matrix Φ_0 and the corresponding balancing sequence will be

$$I, \Phi_0, \Phi_0^2, \ldots, \Phi_0^n, \ldots,$$

where

$$\Phi_0^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}.$$

The roots of the characteristic equation (balancing equation (1.1)) of Q_B are indeed λ_1 and λ_2. Observe that these roots are nothing but the diagonal elements of Φ_0. Thus, (1.1) is the characteristic equation for both Q_B and Φ_0 and by Cayley-Hamilton theorem, each of these matrices satisfy this equation. Comparing (2.1) and (1.1), we have in fact a characterization for all matrices which give rise to balancing sequence of the type (1.2). The following theorem demonstrates the fact:

Theorem 3.3. *A necessary and sufficient condition for the matrix A to be a generator of a balancing sequence (1.2) is that its characteristic equation is the balancing equation.*

The following corollary is an immediate consequences of Theorem 3.3.

Corollary 3.4. *Any two matrix generators of non-trivial balancing sequence of matrices are similar.*

The following corollary immediately follows from Corollary 3.4.

Corollary 3.5. *The matrices Q_B and Φ_0 are similar, that is there exist a non-singular matrix T such that $Q_B = T\Phi_0 T^{-1}$ where the columns of T are eigenvectors of Q_B corresponding to the eigenvalues λ_1 and λ_2.*

Since $Q_B T = T\Phi_0$, a straight forward computation shows that

$$T = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$

It is clear from Corollary 3.4 that $Q_B^n = T\Phi_0^n T^{-1}$ and hence Q_B^n is similar to Φ_0^n. Thus,

$$\det(Q_B^n) = \det(\Phi_0); \quad trace(Q_B^n) = trace(\Phi_0),$$

and we obtain our first pair of balancing identities as follows.

Corollary 3.6. *For every natural number n,*

$$B_n^2 - B_{n+1}B_{n-1} = 1; \;\; B_{n+1} - B_{n-1} = \lambda_1^n + \lambda_2^n.$$

Since the identity $B_{n+1} - B_{n-1} = 2C_n$ is valid for all natural number n [8], we notice that the second part of Corollary 3.6 establishes the proof of the Binet's formula for Lucas-balancing numbers $C_n = \frac{\lambda_1+\lambda_2}{2}$ [7, 8].

Further, motivated from the general term $Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}$ in Example 3.1, we will now examine that whether the sequence with general term $P^n = \begin{pmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{pmatrix}$ is a balancing sequence (1.2). Observe that, for $n = 1$ the matrix $P = \begin{pmatrix} 17 & -3 \\ 3 & -1 \end{pmatrix}$ does not satisfy the balancing equation (1.1). Thus, from Theorem 3.3 we conclude that P^n is not a balancing sequence. Nevertheless, we will show in the next example that the Lucas-balancing numbers, do in fact, enter into the picture in a natural way.

Example 3.7. Setting $y = 1, \; u = 8$ in (2.7), yields $v = 3, \; x = 3$ and we obtain the sequence generator $S = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}$. The corresponding balancing sequence is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}, \begin{pmatrix} 17 & 6 \\ 8\cdot 6 & 17 \end{pmatrix}, \begin{pmatrix} 99 & 35 \\ 8\cdot 35 & 99 \end{pmatrix}, \ldots.$$

The general term of the above sequence can be easily shown as

$$S^n = \begin{pmatrix} C_n & B_n \\ 8B_n & C_n \end{pmatrix}.$$

Similarity of Q_B^n with S^n, by the invariance of trace results

$$2C_n = B_{n+1} - B_{n-1},$$

and by the invariance of determinant results

$$C_n^2 - 8B_n^2 = B_n^2 - B_{n+1}B_{n-1} = 1.$$

Whereas similarity of Φ_0^n with S^n implies by trace gives rise to the Binet's formula for Lucas-balancing numbers $C_n = \frac{\lambda_1^n + \lambda_2^n}{2}$ and by the determinant

$$\lambda_1^n - \lambda_2^n = C_n^2 - 8B_n^2 = 1.$$

Example 3.8. Setting $y = 1,\ u = -1$ in (2.7), one set of solutions is $v = 6,\ x = 0$ and we obtain the matrix $R = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}$. The general form will be

$$R^n = \begin{pmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{pmatrix}.$$

Similarity of R^n with Φ_0^n and Q_B^n gives the results of Corollary 3.6, whereas, similarity with S^n gives the same results as in Example 3.7.

Example 3.9. By taking $y = 1,\ u = -1$ in (2.7), we obtain $v = -1,\ x = 7$ and the generator $F^n = \begin{pmatrix} 7 & 1 \\ -8 & -1 \end{pmatrix}$. The corresponding balancing sequence is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ -8 & -1 \end{pmatrix}, \begin{pmatrix} 41 & 6 \\ -8\cdot 6 & -7 \end{pmatrix}, \begin{pmatrix} 231 & 35 \\ -8\cdot 35 & -41 \end{pmatrix} \ldots,$$

where the general term can be easily shown as

$$L^n = \begin{pmatrix} c_n & B_n \\ -8B_n & -c_{n-1} \end{pmatrix},$$

where b_n denotes the n^{th} cobalancing number and $c_n = \sqrt{8b_n^2 + 8b_n + 1}$ denotes the n^{th} Lucas-cobalancing number [6, 7, 8].

Similarity with Q_B^n, Φ_0^n, S^n respectively give rise to the following identities.

$$\begin{aligned}&c_n c_{n-1} - B_{n+1}B_{n-1} = 7B_n^2,\\&8B_n^2 - c_n c_{n-1} = 1,\\&C_n^2 + c_n c_{n-1} = 16B_n^2.\end{aligned}$$

Whereas similarity with R^n gives same result as Q_B^n.

4 A matrix approach for the proof of Binet's formula for balancing and Lucas-balancing numbers

In this section, we offer two examples of generators of balancing sequences and compute their eigenvalues. With this new tool, we will then establish the Binet's formula for B_n and C_n in terms of λ_1 and λ_2.

Choose $u = 0$ and $y \neq 0$ to be arbitrary in (2.7), we obtain $v = \lambda_1$ and $x = \lambda_2$ and the matrix

$$\Phi_{0y} = \Phi y(say) = \begin{pmatrix}\lambda_1 & y\\ 0 & \lambda_2\end{pmatrix}.$$

The corresponding balancing sequence is

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\lambda_1^2 & 6y\\ 0 & \lambda_2^2\end{pmatrix}, \begin{pmatrix}\lambda_1^3 & 35y\\ 0 & \lambda_2^3\end{pmatrix}, \ldots.$$

The general term of the above sequence can be easily seen as

$$\Phi_y^n = \begin{pmatrix}\lambda_1^n & B_n y\\ 0 & \lambda_2^n\end{pmatrix}.$$

The eigenvectors of the matrix Φ_y corresponding to the eigenvalues λ_1 and λ_2 are respectively $\begin{pmatrix}a\\ 0\end{pmatrix}$ and $\begin{pmatrix}a\\ \frac{\lambda_2-\lambda_1}{y}a\end{pmatrix}$, where $a \neq 0$. Choose $a = 1$. Since $\lambda_2 - \lambda_1 = -2\sqrt{8}$ and taking $y = 2\sqrt{8}$, we have two eigenvectors $\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\begin{pmatrix}1\\ -1\end{pmatrix}$ corresponding to the eigenvalues λ_1 and $\lambda 2$ respectively. Setting $D = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ and using Corollary 3.4, we obtain $\Phi_{2\sqrt{8}} = D\Phi_0 D^{-1}$. It follows that $\Phi_{2\sqrt{8}}^n = D\Phi_0^n D^{-1}$, and hence

$$\Phi_{2\sqrt{8}}^n D = D\Phi_0^n. \tag{4.1}$$

Now (4.1) can be rewritten as

$$\begin{pmatrix} \lambda_1^n & 2\sqrt{8}\lambda_1^n \\ 0 & -\lambda_2^n \end{pmatrix} = \begin{pmatrix} \lambda_1^n & \lambda_2^n \\ 0 & -\lambda_2^n \end{pmatrix}.$$

Equating the elements of first row and second column, we get

$$B_n = \frac{\lambda_1 - \lambda_2}{2\sqrt{8}},$$

which is nothing but the Binet's formula for balancing numbers.

Finally, we present an example where we permit our generator matrix to be complex.

Setting $y = 3,\ u = 3$ in (2.7), we obtain $v = 3 + i,\ x = 3 - i$ and the matrix

$$E = \begin{pmatrix} 3 - i & 3 \\ 3 & 3 + i \end{pmatrix}.$$

The corresponding balancing sequence is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 - i & 3 \\ 3 & 3 + i \end{pmatrix}, \begin{pmatrix} 17 - 6i & 3 \cdot 6 \\ 3 \cdot 6 & 17 + 6i \end{pmatrix}, \ldots,$$

with the general term

$$E^n = \begin{pmatrix} C_n - B_n i & 3B_n \\ 3B_n & C_n + B_n i \end{pmatrix}.$$

Proceeding as in the previous example, we obtain the eigenvectors corresponding to the eigenvalues λ_1 and λ_2 respectively as $\begin{pmatrix} \frac{1}{3}[\lambda_1 - (3+i)] \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{3}[\lambda_2 - (3+i)] \\ 1 \end{pmatrix}$. We construct the matrix

$$M = \begin{pmatrix} \frac{1}{3}[\lambda_1 - (3+i)] & \frac{1}{3}[\lambda_2 - (3+i)] \\ 1 & 1 \end{pmatrix}$$

and using Corollary 3.4, we obtain

$$E^n M = M\Phi_0^n,$$

which on simplification and comparison of second row and first column entries of the matrix gives

$$B_n[\lambda_1 - (3+i)] + C_n + iB_n = \lambda_1^n.$$

Further simplification results

$$\lambda_1^n = (\lambda_1 - 3)B_n + C_n,$$

that is

$$\lambda_1^n = \sqrt{8}B_n + C_n. \tag{4.2}$$

Similarly we can obtain

$$\lambda_2^n = -\sqrt{8}B_n + C_n. \tag{4.3}$$

Adding (4.2) and(4.3), we obtain

$$C_n = \frac{\lambda_1 + \lambda_2}{2},$$

which is the Binet's formula of Lucas-balancing numbers.

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