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Restricted super line signed graph $\mathcal{RL}_r(S)$

P. Siva Kota Reddy[1] and U. K. Misra[2]

[1] Department of Mathematics
Siddaganga Institute of Technology
B. H. Road, Tumkur–572 103, India
e-mails: reddy_math@yahoo.com,
pskreddy@sit.ac.in

[2] Department of Mathematics
Berhampur University
Berhampur–760 007, Orissa, India

Dedicated to Honorable Shri Dr. M. N. Channabasappa
on his 82nd birthday

Abstract: A *signed graph (marked graph)* is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ ($\mu : V \to \{+, -\}$) is a function. The *restricted super line graph of index* r of a graph G, denoted by $\mathcal{RL}_r(G)$. The vertices of $\mathcal{RL}_r(G)$ are the r-subsets of $E(G)$ and two vertices $P = \{p_1, p_2, ..., p_r\}$ and $Q = \{q_1, q_2, ..., q_r\}$ are adjacent if there exists exactly one pair of edges, say p_i and q_j, where $1 \leq i, j \leq r$, that are adjacent edges in G.

Analogously, one can define the *restricted super line signed graph of index* r of a signed graph $S = (G, \sigma)$ as a signed graph $\mathcal{RL}_r(S) = (\mathcal{RL}_r(G), \sigma')$, where $\mathcal{RL}_r(G)$ is the underlying graph of $\mathcal{RL}_r(S)$, where for any edge PQ in $\mathcal{RL}_r(S)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$. It is shown that for any signed graph S, its $\mathcal{RL}_r(S)$ is balanced and we offer a structural characterization of restricted super line signed graphs of index r.

Further, we characterize signed graphs S for which $\mathcal{RL}_r(S) \sim \mathcal{L}_r(S)$ and $\mathcal{RL}_r(S) \cong \mathcal{L}_r(S)$, where $\sim$ and $\cong$ denotes switching equivalence and isomorphism and $\mathcal{RL}_r(S)$ and $\mathcal{L}_r(S)$ are denotes the restricted super line signed graph of index r and super line signed graph of index r of S, respectively.

Keywords: Signed graphs, Marked graphs, Balance, Switching, Restricted super line signed graph, Super line signed graphs, Negation.
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1 Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [7]. We consider only finite, simple graphs free from self-loops.

Cartwright and Harary [4] considered graphs in which vertices represent persons and the edges represent symmetric dyadic relations amongst persons each of which designated as being positive or negative according to whether the nature of the relationship is positive (friendly, like, etc.) or negative (hostile, dislike, etc.). Such a network S is called a *signed graph* (Chartrand [5]; Harary et al. [10]).

Signed graphs are much studied in literature because of their extensive use in modeling a variety socio-psychological process (e.g., see Katai and Iwai [12], Roberts [15] and Roberts and Xu [16]) and also because of their interesting connections with many classical mathematical systems (Zaslavsky [36]).

A cycle in a signed graph S is said to be *positive* if the product of signs of its edges is positive. A cycle which is not positive is said to be *negative*. A signed graph is then said to be *balanced* if every cycle in it is positive (Harary [8]). Harary and Kabell [11] developed a simple algorithm to detect balance in signed graphs as also enumerated them.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S. In a signed graph $S = (G, \sigma)$, for any $A \subseteq E(G)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

The following characterization of balanced signed graphs is well known.

Proposition 1. (**E. Sampathkumar [17]**) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [36].

Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs (See also [14, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]). The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$.

Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [34]) or *cycle isomorphic* (see [35]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known:

Proposition 2. (**T. Zaslavsky** [35]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2 Restricted super line signed graph $\mathcal{L}_r(S)$

In [13], K. Manjula introduced the concept of the *restricted super line graph*, which generalizes the notion of line graph. For a given G, its restricted super line graph $\mathcal{RL}_r(G)$ of index r is the graph whose vertices are the r-subsets of $E(G)$, and two vertices $P = \{p_1, p_2, ..., p_r\}$ and $Q = \{q_1, q_2, ..., q_r\}$ are adjacent if there exists exactly one pair of edges, say p_i and q_j, where $1 \leq i, j \leq r$, that are adjacent edges in G. In [2], the authors introduced the concept of the *super line graph* as follows: For a given G, its super line graph $\mathcal{L}_r(G)$ of index r is the graph whose vertices are the r-subsets of $E(G)$, and two vertices P and Q are adjacent if there exist $p \in P$ and $q \in Q$ such that p and q are adjacent edges in G. Clearly $\mathcal{RL}_r(G)$ is a spanning subgraph of $\mathcal{L}_r(G)$. From the definitions of $\mathcal{RL}_r(G)$ and $\mathcal{L}_r(G)$, it turns out that $\mathcal{RL}_1(G)$ and $\mathcal{L}_1(G)$ coincides with the line graph $L(G)$.

In this paper, we extend the notion of $\mathcal{RL}_r(G)$ to realm of signed graphs as follows: The *restricted super line signed graph of index r* of a signed graph $S = (G, \sigma)$ as a signed graph $\mathcal{RL}_r(S) = (\mathcal{RL}_r(G), \sigma')$, where $\mathcal{RL}_r(G)$ is the underlying graph of $\mathcal{RL}_r(S)$, where for any edge PQ in $\mathcal{RL}_r(S)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$.

Hence, we shall call a given signed graph S is a *restricted super line signed graph of index r* if it is isomorphic to the restricted super line signed graph of index r, $\mathcal{RL}_r(S')$ of some signed graph S'. In the following subsection, we shall present a characterization of restricted super line signed graph of index r.

The following result indicates the limitations of the notion $\mathcal{RL}_r(S)$ as introduced above, since the entire class of unbalanced signed graphs is forbidden to be restricted super line signed graphs of index r.

Proposition 3. *For any signed graph $S = (G, \sigma)$, its $\mathcal{RL}_r(S)$ is balanced.*

Proof. Let $\sigma^{'}$ denote the signing of $\mathcal{RL}_r(S)$ and let the signing σ of S be treated as a marking of the vertices of $\mathcal{RL}_r(S)$. Then by definition of $\mathcal{RL}_r(S)$ we see that $\sigma'(P, Q) = \sigma(P)\sigma(Q)$, for every edge PQ of $\mathcal{RL}_r(S)$ and hence, by Proposition 1, the result follows. □

For any positive integer k, the k^{th} iterated restricted super line signed graph of index r, $\mathcal{RL}_r(S)$ of S is defined as follows:

$$\mathcal{RL}_r^0(S) = S, \mathcal{RL}_r^k(S) = \mathcal{RL}_r(\mathcal{RL}_r^{k-1}(S))$$

Corollary 4. *For any signed graph $S = (G, \sigma)$ and any positive integer k, $\mathcal{RL}_r^k(S)$ is balanced.*

In [26], the authors introduced the notion of the *super line signed graph*, which generalizes the notion of line signed graph [6]. The *super line signed graph of index* r of a signed graph $S = (G, \sigma)$ as a signed graph $\mathcal{L}_r(S) = (\mathcal{L}_r(G), \sigma')$, where $\mathcal{L}_r(G)$ is the underlying graph of $\mathcal{L}_r(S)$, where for any edge PQ in $\mathcal{L}_r(S)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$. The above notion restricted super line signed graph is another generalization of line signed graphs.

Proposition 5. **(P.S.K.Reddy and S. Vijay** [26]) *For any signed graph $S = (G, \sigma)$, its $\mathcal{L}_r(S)$ is balanced.*

In [13], the author characterized whose restricted super line graphs of index r that are isomorphic to $\mathcal{L}_r(S)$.

Proposition 6. **(K. Manjula** [13]) *For a graph $G = (V, E)$, $\mathcal{RL}_r(G) \cong \mathcal{L}_r(G)$ if, and only if, G is either $K_{1,2} \cup nK_2$ or nK_2.*

We now characterize signed graphs those $\mathcal{RL}_r(S)$ are switching equivalent to their $\mathcal{L}_r(S)$.

Proposition 7. *For any signed graph $S = (G, \sigma)$, $\mathcal{RL}_r(S) \sim \mathcal{L}_r(S)$ if, and only if, G is either $K_{1,2} \cup nK_2$ or nK_2.*

Proof. Suppose $\mathcal{RL}_r(S) \sim \mathcal{L}_r(S)$. This implies, $\mathcal{RL}_r(G) \cong \mathcal{L}_r(G)$ and hence by Proposition 6, we see that the graph G must be isomorphic to either $K_{1,2} \cup nK_2$ or nK_2.

Conversely, suppose that G is either $K_{1,2} \cup nK_2$ or nK_2. Then $\mathcal{RL}_r(G) \cong \mathcal{L}_r(G)$ by Proposition 6. Now, if S any signed graph on any of these graphs, by Proposition 3 and Proposition 5, $\mathcal{RL}_r(S)$ and $\mathcal{L}_r(S)$ are balanced and hence, the result follows from Proposition 2. □

We now characterize signed graphs those $\mathcal{RL}_r(S)$ are isomorphic to their $\mathcal{L}_r(S)$. The following result is a stronger form of the above result.

Proposition 8. *For any signed graph $S = (G, \sigma)$, $\mathcal{RL}_r(S) \cong \mathcal{L}_r(S)$ if, and only if, G is either $K_{1,2} \cup nK_2$ or nK_2.*

Proof. Clearly $\mathcal{RL}_r(S) \cong \mathcal{L}_r(S)$, where G is either $K_{1,2} \cup nK_2$ or nK_2. Consider the map $f : V(\mathcal{RL}_r(G)) \to V(\mathcal{L}_r(S))$ defined by $f(e_1e_2, e_2e_3) = (e_1'e_2', e_2'e_3')$ is an isomorphism. Let σ be any signing on $K_{1,2} \cup nK_2$ or nK_2. Let $e = (e_1e_2, e_2e_3)$ be an edge in $\mathcal{RL}_r(G)$, where G is $K_{1,2} \cup nK_2$ or nK_2. Then sign of the edge e in $\mathcal{RL}_r(G)$ is the $\sigma(e_1e_2)\sigma(e_2e_3)$ which is the sign of the edge $(e_1'e_2', e_2'e_3')$ in $\mathcal{L}_r(G)$, where G is $K_{1,2} \cup nK_2$ or nK_2. Hence the map f is also a signed graph isomorphism between $\mathcal{RL}_r(S)$ and $\mathcal{L}_r(S)$. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [9] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{RL}_r(S)$ is balanced (Proposition 3). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{RL}_r(S)$ is balanced.

Proposition 9. *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{RL}_r(G)$ is bipartite then $\eta(\mathcal{RL}_r(S))$ is balanced.*

Proof. Since, by Proposition 3, $\mathcal{RL}_r(S)$ is balanced, if each cycle C in $\mathcal{RL}_r(S)$ contains even number of negative edges. Also, since $\mathcal{RL}_r(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{RL}_r(S)$ is also even. Hence $\eta(\mathcal{RL}_r(S))$ is balanced. □

2.1 Characterization of restricted super line signed graphs $\mathcal{RL}_r(S)$

The following result characterize signed graphs which are restricted super line signed graphs of index r.

Proposition 10. *A signed graph $S = (G, \sigma)$ is a restricted super line signed graph of index r if and only if S is balanced signed graph and its underlying graph G is a restricted super line graph of index r.*

Proof. Suppose that S is balanced and G is a $\mathcal{RL}_r(G)$. Then there exists a graph H such that $\mathcal{L}_r(H) \cong G$. Since S is balanced, by Proposition 1, there exists a marking μ of G such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{RL}_r(S') \cong S$. Hence S is a restricted super line signed graph of index r.

Conversely, suppose that $S = (G, \sigma)$ is a restricted super line signed graph of index r. Then there exists a signed graph $S' = (H, \sigma')$ such that $\mathcal{RL}_r(S') \cong S$. Hence G is the $\mathcal{RL}_r(G)$ of H and by Proposition 3, S is balanced. □

If we take $r = 1$ in $\mathcal{RL}_r(S)$, then this is the ordinary line signed graph. In [20, 21], the authors obtained structural characterization of line signed graphs and line signed digraphs and clearly Proposition 10 is the generalization of line signed graphs.

Proposition 11. (**E. Sampathkumar et al. [20]**) *A signed graph $S = (G, \sigma)$ is a line signed graph if, and only if, S is balanced signed graph and its underlying graph G is a line graph.*

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