On generalized multiplicative perfect numbers

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Abstract: In this paper we define $T^*T$ multiplicative divisors function. This notion leads us to generalized multiplicative perfect numbers like $T^*T$ perfect numbers, $k − T^*T$ perfect numbers and $T^*_0T$−super-perfect numbers. We attempt to characterize these numbers.

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1 Introduction

A natural number $n$ is said to be perfect number if it is equal to the sum of its proper divisors. If $\sigma$ denotes the sum of divisors, for any perfect number $n$, $\sigma(n) = 2n$. The Euclid–Euler theorem gives the form of even perfect numbers in the form $n = 2^p − 1(2^p − 1)$, where $2^p − 1$ is a Mersenne prime. Moreover $n$ is said to be super-perfect if $\sigma(\sigma(n)) = 2n$. The Suryanarayana–Kanold theorem gives the general form of even super-perfect numbers $−n = 2^k$, where $2^k + 1$ is prime. No odd super-perfect numbers are known. For new proofs of these results, see [5, 9]. A divisor $d$ of a natural number $n$ is said to be unitary divisor if $d^n = 1$, and $n$ is unitary perfect if $\sigma^*(n) = 2n$, where $\sigma^*$ denotes the sum of unitary divisors of $n$. The notion of unitary perfect numbers was introduced M. V. Subbarao and L. J. Waren in 1966, [8]. Five unitary even perfect numbers are known and it is true that no unitary perfect numbers of the form $2^m$ are where $s$ is a square free odd integer [3]. Sándor in [6] introduced the concept of multiplicatively divisor function $T(n)$ and multiplicatively perfect and super-perfect numbers and characterized them. If $T(n)$ denote the product of all divisors of $n$, then

$$T(n) = \prod_{d|n} d = n^{\tau(n)/2},$$

where $\tau(n)$ is the number of divisors of $n$. The number $n > 1$ is multiplicatively perfect (or shortly m-perfect) if $T(n) = n^2$, and multiplicatively super-perfect (m-super-perfect), if $T(T(n)) = n^2$. In [1], Antal Bege introduced the concept of unitary divisor function $T^*(n)$ and...
unitary perfect and super-perfect numbers and characterized them multiplicatively. Let $T^*(n)$ denote the product of all unitary divisors of $n$:

$$T^*(n) = \prod_{d|n} d = n^{\tau^*(n)/2},$$

where $\left(d, \frac{n}{d}\right) = 1$ and $\tau^*(n)$ is the number of unitary divisors of $n$. The number $n > 1$ is multiplicatively unitary perfect (or shortly m-unitary-perfect) if $T^*(n) = n^2$, and multiplicatively unitary super-perfect (m-unitary-super-perfect), if $T^*(T^*(n)) = n^2$. It is to be noted that there are no m-super-perfect and m-unitary-super perfect numbers.

## 2 $T^*T$-perfect numbers

**Definition 2.** Let $[T^*T](n)$ or $[TT^*](n)$ denote the product of $T(n)$ and $T^*(n)$, i.e. $[T^*T](n) = T^*(n)T(n)$. Let us call the number $n > 1$ as $T^*T$–perfect number if $[T^*T](n) = n^2$.

**Theorem 2.1.** For $n > 1$ there are no $T^*T$–perfect numbers for non-prime $n$.

**Proof:** Let $n = p_1^{\alpha_1}p_2^{\alpha_2}......p_r^{\alpha_r}$ be the prime factorisation of $n > 1$. It is well-known that

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_r + 1) \tag{2.1}$$

and

$$\tau^*(n) = 2^\omega(n) = 2^r, \tag{2.2}$$

where $\omega(n)$ is the number of distinct prime divisors of $n$.

$$[T^*T](n) = T^*(n)T(n) = n^{\frac{\tau(n)}{2}}n^{\frac{\tau^*(n)}{2}} = n^{\frac{\tau(n)\tau^*(n)}{2}}$$

For $T^*T$–perfect numbers

$$2^r + (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_r + 1) = 4.$$

Since $r \geq 1$, we can have only $(\alpha_i + 1) = 2$ and $r = 1$, giving $n = p_1$. There are no other solutions $n > 1$ ($n = 1$ is a trivial solution) of the equation.

Thus primes are $T^*T$–perfect numbers.

For any $n \geq 2$ we have $\tau(n) \geq 2$, so $T(n) \geq 2$.

If $n$ is not a prime, then it is immediate that $\tau(n) \geq 3$, giving

$$T(n) \geq n^{\frac{3}{2}} \tag{2.3}$$

If $n$ is not a prime, then

$$T^*(n) \geq n \tag{2.4}$$

Now relations (2.3) and (2.4) together give $[T^*T](n) \geq n^{\frac{5}{2}}$, where $n$ is not a prime.

Thus, by $\frac{5}{2} > 2$, there are no $T^*T$–perfect number for non prime $n$.

**Corollary 2.2.** Perfect numbers are not $T^*T$–perfect numbers.
3 $k-T^*T$–perfect numbers

In a similar manner, one can define $k-T^*T$–perfect numbers by $[T^*T](n) = n^k$, where $k \geq 2$ is given. Since the equation $2^r + (\alpha_1 + 1)(\alpha_2 + 1)…(\alpha_r + 1) = 2k$ has a finite number of solutions, the general form of $k-T^*T$–perfect numbers can be determined. We present certain particular cases in the following result.

Theorem 3.1.

(i) All tri-$T^*T$–perfect numbers have the form $n = p_1^3$;
(ii) All 4–$T^*T$–perfect numbers have the form $n = p_1p_2$ or $n = p_1^5$;
(iii) All 5–$T^*T$–perfect numbers have the form $n = p_2^2 p_2$ or $n = p_1^7$;
(iv) All 6–$T^*T$–perfect numbers have the form $n = p_3 p_2$ or $n = p_1^9$;
(v) All 7–$T^*T$–perfect numbers have the form $n = p_1^4 p_2$ or $n = p_1^{11}$;
(vi) All 8–$T^*T$–perfect numbers have the form $n = p_1p_2p_3$ or $n = p_1^6 p_2$ or $n = p_1^3 p_2^2$ or $n = p_1^{13}$;
(vii) All 9–$T^*T$–perfect numbers have the form $n = p_1^6 p_2$ or $n = p_1^{15}$;
(viii) All 10–$T^*T$–perfect numbers have the form $n = p_1^2 p_2^2 p_3$ or $n = p_1^7 p_2$ or $n = p_1^3 p_2^3$ or $n = p_1^{17}$;
(ix) All 11–$T^*T$–perfect numbers have the form $n = p_1^5 p_2^2$ or $n = p_1^8 p_2$ or $n = p_1^{19}$;
(x) All 12–$T^*T$–perfect numbers have the form $n = p_1^3 p_2 p_3$ or $n = p_1^9 p_2$ or $n = p_1^4 p_2^3$ or $n = p_1^{21}$, etc.

Here $p_i$ denote certain distinct primes. We prove only the cases (vi) and (x).

Proof: (vi) For the 8–$T^*T$–perfect number $n$, $[T^*T](n) = n^8$, so we must solve the equation

$2^r + (\alpha_1 + 1)(\alpha_2 + 1)…(\alpha_r + 1) = 16$

in $\alpha_r$ and $r$. It is easy to see that the following four cases are possible:

(I) $r = 1$, $\alpha_1 + 1 = 14$;

(II) $r = 2$, $\alpha_1 + 1 = 4, \alpha_2 + 1 = 3$;

(III) $r = 2$, $\alpha_1 + 1 = 6, \alpha_2 + 1 = 2$;

(IV) $r = 3$, $\alpha_1 + 1 = 2, \alpha_2 + 1 = 2, \alpha_3 + 1 = 2$.

This gives the general forms of all 8–$T^*T$–perfect numbers, namely:

($r = 1, \alpha_1 = 13) n = p_1^{13}$;

($r = 2, \alpha_1 = 3, \alpha_2 = 2) n = p_1^3 p_2^2$;
(r = 2, α_1 = 5, α_2 = 1) n = p_i^5 p_2; \\
(r = 3, α_1 = 1, α_2 = 1, α_3 = 1) n = p_i p_2 p_3.

(x) To find the general form of 12−T*T−perfect numbers, we must solve the equation

\[ 2^r + (α_1 + 1)(α_2 + 1)\ldots(α_r + 1) = 24 \]

in \( α_r \) and \( r \). It is easy to see that the following four cases are possible:

(I) \( r = 1, α_1 + 1 = 21; \)

(II) \( r = 2, α_1 + 1 = 4, α_2 + 1 = 5; \)

(III) \( r = 2, α_1 + 1 = 10, α_2 + 1 = 2; \)

(IV) \( r = 3, α_1 + 1 = 3, α_2 + 1 = 2, α_3 + 1 = 2. \)

Thus the general forms of all 12−T*T−perfect numbers are namely:

\[
\begin{align*}
(r = 1, α_1 = 21) & \quad n = p_i^{21} ; \\
(r = 2, α_1 = 3, α_2 = 4) & \quad n = p_i^3 p_2^4 ; \\
(r = 2, α_1 = 9, α_2 = 1) & \quad n = p_i^9 p_2 ; \\
(r = 3, α_1 = 2, α_2 = 1, α_3 = 1) & \quad n = p_i^3 p_2 p_3. \\
\end{align*}
\]

Corollary 3.2. (i) There are no perfect numbers which are tri−T*T−perfect number. 
(ii) \( n = 6 \) is the only perfect number which is 4−T*T−perfect number. 
(iii) \( n = 28 \) is the only perfect number which is 5−T*T−perfect number. 
(iv) \( n = 496 \) is the only perfect number which is 7−T*T−perfect number. 
(v) \( n = 8128 \) is the only perfect number which is 9−T*T−perfect number.

Theorem 3.3. Let \( p \) be a prime, with \( 2^p - 1 \) being a Mersenne prime. Then \( n = 2^{p-1}(2^p - 1) \) is the only perfect number which is a \( (p + 2)−T*T−perfect number. \)

**Proof:** If \( n = 2^{p-1}(2^p - 1) \) is an even perfect number, then \( τ(n) = 2p, \ )\( ow(n) = 2, \ )\( τ*(n) = 4, \) and so

\[ [T*T](n) = n^{\frac{τ(n)×τ*(n)}{2}} = n^{\frac{2p\times4}{2}} = n^{p+2}. \]

4 \( T_*0T−super-perfect and k−T_*0T−perfect numbers \)

**Definition 4.1:** The number \( n > 1 \) is a \( T_*0T−super-perfect number if \( T^*(T(n)) = n^2 ; \) and \( k−T_*0T−perfect number if \( T^*(T(n)) = n^k ; \) where \( k \geq 3. \)

**Theorem 4.2.** All \( T_*0T−super-perfect numbers have the form \( n = p_i^3, \) where \( p_i \) is an arbitrary prime.

**Proof:** First, we determine \( T^*(T(n)) : \)
\[
T^*(T(n)) = \left( T(n) \right)^\frac{\tau(T(n))}{2} = \left( n^\frac{\tau(n)}{2} \right)^\frac{\tau(T(n))}{2}
\]  \hspace{2cm} (4.1)

\[
\tau^*(T(n)) = \tau^*(n^\frac{\tau(n)}{2}) = \tau^*(n)
\]  \hspace{2cm} (4.2)

From (4.1) and (4.2), \( T^*(T(n)) = n^{\frac{\tau(n)}{4}} \). By using the relations (2.1) and (2.2), for \( T^* \)-super-perfect numbers

\[
2^r (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 8.
\]

Since \( r \geq 1 \), we can have only \( \alpha_1 + 1 = 4 \) and \( r = 1 \), implying \( r = 1, \alpha_1 = 3 \), i.e. \( n = p_1^3 \). In a similar manner \( k-T^* \)-perfect numbers can be defined. Since the equation

\[
2^r (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 4k
\]

has a finite number of solutions, the general form of \( k-T^* \)-perfect numbers can be determined.

**Theorem 4.3.**

(i) All tri-\(T^* \)-perfect numbers have the form \( n = p_1^5 \);

(ii) All 4-\(T^* \)-perfect numbers have the form \( n = p_1 p_2 \) or \( n = p_1^7 \);

(iii) All 5-\(T^* \)-perfect numbers have the form \( n = p_1^9 \);

(iv) All 6-\(T^* \)-perfect numbers have the form \( n = p_1^2 p_2 \) or \( n = p_1^{11} \);

(v) All 7-\(T^* \)-perfect numbers have the form \( n = p_1^{13} \);

(vi) All 8-\(T^* \)-perfect numbers have the form \( n = p_1^3 p_2 \) or \( n = p_1^{15} \);

(vii) All 9-\(T^* \)-perfect numbers have the form \( n = p_1^2 p_2^2 \) or \( n = p_1^{17} \);

(viii) All 10-\(T^* \)-perfect numbers have the form \( n = p_1^4 p_2 \) or \( n = p_1^{19} \);

**Proof:** We prove only the case (viii). For 10-\(T^* \)-perfect number \( T^*(T(n)) = n^{10} \). We must solve the equation

\[
2^r (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 40
\]

in \( r \) and \( \alpha_r \). It is easy to see that the following cases are possible:

(I) \( r = 1, \alpha_1 + 1 = 20 \).

(II) \( r = 2, \alpha_1 + 1 = 5, \alpha_2 + 1 = 2 \).

This gives the general form of all 10-\(T^* \)-perfect numbers, namely:

\( (r = 2, \alpha_1 = 4, \alpha_2 = 1) \) \( n = p_1^4 p_2 \);

\( (r = 1, \alpha_1 = 19) \) \( n = p_1^{19} \). □

**Theorem 4.4.** Let \( p \) be a prime, with \( 2^p - 1 \) being a Mersenne prime. Then \( 2^p - 1 \) is the only perfect number, which is \( 2p-T^* \)-perfect number.

**Proof:** By writing \( 2^r (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 8p \) (where \( p \) is prime), the following cases are only possible:
(i) $r = 2, \alpha_1 + 1 = 2$ , $\alpha_2 + 1 = p$

(ii) $r = 1$, $\alpha_1 + 1 = 4p$

Then $n = p_1p_2^{p-1}$ or $n = p_1^{4p-1}$ are the general form of $2p-T^*_{0}T$–perfect numbers. By the Euler–Euclid theorem, $p_1p_2^{p-1} = 2^{p-1}(2^p - 1)$ iff $p_1 = 2^p - 1$ and $p_2 = 2$.

5 $k$–$T^*_0T$–perfect numbers

**Definition 5.1.** The number $n > 1$ is a $k$–$T_0T^*$–perfect number (where $k \geq 2$) if $T(T^*(n)) = n^k$.

First, we determine $T(T^*(n))$. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$ be the prime factorisation of $n > 1$, then $\tau^*(n) = 2^r$ and $T^*(n) = n^{2^{r-1}}$.

$$T(T^*(n)) = (T^*(n))^{\frac{\tau^*(n)}{2}} = \left( n^{2^{r-1}} \right)^{\frac{\tau(n)}{2}} = n^{\frac{\tau(n)}{4}}$$ (5.1)

Since $n^{2^{r-1}} = p_1^{\alpha_1}2^{r-1}p_2^{\alpha_2}2^{r-1}...p_r^{\alpha_r}2^{r-1}$ and $\tau(n)$ is a multiplicative function, so

$$\tau(p_i^{\alpha_i2^{r-1}}) = \alpha_i2^{r-1} + 1; \ i = 1, 2, ..., r$$ (5.2)

From the relations (5.1) and (5.2) for $k$–$T_0T^*$–perfect number

$$2^r (\alpha_12^{r-1} + 1)(\alpha_22^{r-1} + 1)...(\alpha_r2^{r-1} + 1) = 4k$$ (5.3)

Solving the equation (5.3) in $r$ and $\alpha_r$, we can determine forms of the $k$–$T_0T^*$–perfect numbers.

**References**


