Notes on Number Theory and Discrete Mathematics
Vol. 19, 2013, No. 4, 4–15

b-Parts and finite b-representation of real numbers

M. H. Hooshmand

Department of Mathematics, Shiraz Branch,
Islamic Azad University, Shiraz, Iran
e-mails: hadi.hooshmand@gmail.com,
hooshmand@iaushiraz.ac.ir

Abstract: The b-parts of real numbers and the generalized division algorithm were considered and discussed in [3]. Also some of their algebraic properties have been studied in [4]. In this paper we continue it and introduce a unique finite representation of real numbers to the base of an arbitrary real number $b \neq 0, \pm 1$ (namely finite b-representation), by using them. Finally we prove a necessary and sufficient conditions for the finite b-representation to be digital.
Keywords: b-integer part, b-decimal part, generalized division algorithm, radix representation and expansion of real numbers, b-digital sequence.
AMS Classification: 11A63, 11A67.

1 Preliminaries

For any real number a denote by $[a]$ the largest integer not exceeding a and put $(a) = a - [a]$ (the decimal part of a). Now let b be a nonzero constant real number. For all real numbers a set

$$[a]_b = b[\frac{a}{b}] \ , \ (a)_b = b(\frac{a}{b}).$$

We call the notation $[a]_b$ *b-integer part of a* and $(a)_b$ *b-decimal part of a*. Also $[a]_b$ and $(a)_b$ are called *b-parts of a*.
Clearly $a = [a]_b + (a)_b$ where

$$[a]_b \in b\mathbb{Z} =< b >, \ \ (a)_b \in \mathbb{R}_b := b[0,1) = \{bd | 0 \leq d < 1\}.$$

Since $(a)_1 = (a)$ to prevent any confusion between decimal and parentheses notation, sometimes we use the symbol $(a)_1$ instead of (a).
It is easy to see that the following properties (I)-(IV) hold:

(I) For every $\beta \in b\mathbb{Z}$, we have $[a+\beta]_b = [a]_b + \beta,\ (a+\beta)_b = (a)_b$ so if m, n are integers, then

$$(ma+nc)_b = (m(a)_b + nc)_b = (ma + n(c)_b)_b = (m(a)_b + n(c)_b)_b = ((ma+nc)_b)_b.$$

Therefore the b-decimal and b-integer part functions $(x)_b$ and $[x]_b$ are idempotent, their compositions are zero and $(x)_b$ satisfies the following functional equations

$$f(f(x)+y-f(y)) = f(x)\ ,\ f(x+y-f(y)) = f(x)\ ,\ f(x+f(y+z)) = f(f(x+y)+z).$$

Note: One can see the general solution of these functional equations in [5]. In fact the above basic properties have led us to a type of functions on groups.

(II)

$$(a)_b = a \Longleftrightarrow a \in \mathbb{R}_b \Longleftrightarrow [a]_b = 0\ \ ,\ \ (a)_b = 0 \Longleftrightarrow a \in b\mathbb{Z} \Longleftrightarrow [a]_b = a.$$

(III)

$$|(a)_b| < |b|\ ,\ |a| - |b| < |[a]_b|\ ,\ \frac{[a]_b}{\text{sgn}(b)} \leq \frac{a}{\text{sgn}(b)} < \frac{[a]_b + b}{\text{sgn}(b)},$$

where sgn is the signum function.

Now applying the above elementary properties we can deduce and state the followings interesting number theoretic explanation of b-parts of real numbers.

(IV) (Number theoretic explanation of b-parts):
For every positive integer b and real a, $[a]_b$ is the same unique integer of the residue class $\{[a]-b+1, \cdots, [a]\}$ (mode b) that is divisible by b (because $b|[a]_b$ and $[a]-b+1 \leq [a]_b \leq [a]$). Also, for the general explanation of $[a]_b$, if $b > 0$, then $[a]_b$ is the largest element of $b\mathbb{Z}$ not exceeding a and if $b < 0$, then $[a]_b$ is the least element of $b\mathbb{Z}$ not less than a.
Now let a , b are positive integers. By the division algorithm we have $a = bq + r$ where q , r are integers and $0 \leq r < b$, so

$$(a)_b = (bq + r)_b = (r)_b = r.$$

It means that $(a)_b$ is the same remainder of the division of a by b. It is an important fact that leads us to the generalized division algorithm (for real numbers) and algebraic properties of b-parts.

Theorem 1.0. Suppose $b \neq 0$ be a fixed real number.
(a) (The unique representation of real numbers by b-parts) For every real number a there exist unique numbers a_1 and a_2 such that

$$a = a_1 + a_2\ \ ,\ \ a_1 \in b\mathbb{Z}\ \ ,\ \ a_2 \in \mathbb{R}_b.$$

(b) (The generalized division algorithm) For every real number a, there exist a unique integer q

and a unique non negative real number r such that

$$a = bq + r \quad , \quad 0 \leq r < |b|.$$

(q and r are called integer quotient and b-bounded remainder of the division of a by b, respectively.)

Proof. See [3] and [4] for two different proofs.

Now applying the above theorem we can here state the general number theoretic explanation of $(a)_b$:
If $b > 0$, then $(a)_b$ is the same b-bounded remainder of the (generalized) division of a by b, and if $b < 0$, then $(a)_b$ is the inverse of the remainder of the division of $-a$ by $-b$ (because $(a)_b = -(-a)_{-b}$).
Therefore $a \equiv c$ (mod b) if and only if $(a)_b = (c)_b$.

(V) If b is a positive integer, then for every real number a we have

$$([a])_b = [(a)_b] = (a)_b - (a) = (a)_b - ((a)_b) = [a] - [[a]]_b = [a] - [a]_b.$$

Because $a = [a]_b + (a)_b = [[a]]_b + ([a])_b + (a)$ and since $b \in \mathbb{Z}^+$, then $([a])_b \in \mathbb{Z}$ and so $0 \leq ([a])_b \leq b-1$ hence $0 \leq ([a])_b + (a) < b$ therefore Theorem 1.0(a) (the unique representation of real numbers by b-parts) implies $(a)_b = ([a])_b + (a)$. On the other hand

$$[a] = [[a]]_b + ([a])_b = [[a]_b + (a)_b] = [a]_b + [(a)_b].$$

Now we can deduce the identities.

(VI) For every real numbers a and $b \neq 0$, the set $\{(na)_b | n \in \mathbb{Z}\}$ is finite if and only if $a \in b\mathbb{Q}$ (i.e. $\frac{a}{b}$ is rational number). In addition if $\frac{a}{b}$ is irrational, then the sequence $(na)_b$ is dense in the close interval $b[0,1]$ ($= [0,b]$ or $[b,0]$).

Because if m and n are two distinct integers, then $(na)_b = (ma)_b$ if and only if $a = \frac{[na]_b - [ma]_b}{n-m}$ (notice that $[na]_b - [ma]_b \in b\mathbb{Z}$). Also if n_0 is a fixed integer and $a = \frac{m}{n_0} b$, then $(n_0 a)_b = (ma)_b = 0$ and for every integer k we have

$$(ka)_b = ([k]_{n_0} a + (k)_{n_0} a)_b = ([\frac{k}{n_0}] n_0 a + (k)_{n_0} a)_b = ([\frac{k}{n_0}](n_0 a)_b + (k)_{n_0} a)_b$$

$$= ((k)_{n_0} a)_b \in \{0, (a)_b, (2a)_b, \cdots, ((n_0 - 1)a)_b\}.$$

In fact we have $\{(na)_b | n \in \mathbb{Z}\} = \{0, (a)_b, (2a)_b, \cdots, ((n_0 - 1)a)_b\}$.

Also the identity $(na)_b = b(n\frac{a}{b})_1$ and the Kronecker's theorem imply the sequence $\{(na)_b\}_{n \geq 1}$ is dense in the close interval $b[0,1]$, if $\frac{a}{b}$ is irrational.

Remark 1.1 As we can see in [4], in fact the set $\{(na)_b | n \in \mathbb{Z}\}$ is a cyclic subgroup of the b-bounded group $(\mathbb{R}_b, +_b)$ (the least real residues group modulo b, as a generalization of the group $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$), where $+_b$ is the b-addition ($x +_b y = (x+y)_b$, $\forall x, y \in \mathbb{R}$). The above property states that a cyclic subgroup of $(b[0,1), +_b)$, generated by a, is dense in $b[0,1]$ if and only if $\frac{a}{b}$ is irrational. Also if $\frac{a}{b} = \frac{m_0}{n_0}$ is a rational number for which $n_0 > 0$, $\gcd(m_0, n_0) = 1$, then the cyclic group $< a >$ is finite and

$$< a >= \{0, (a)_b, (2a)_b, \cdots, ((n_0-1)a)_b\},$$

If a and b are integers, then $(a)_b$ is also an integer. Hence this question has been introduced that when is $(a)_b$ an integer?. The answer of this question is important, because first we want to know that if $a, b \in \mathbb{R}$ and $b > 0$, then when the remainder of the division of a by b is an integer (like the quotient of the division) . Secondly we need it (in the next section) to determine that when the finite b-representation of a real number is digital. Before of stating the related lemma notice that:

A necessary condition for $(a)_b$ to be an integer is that $a \in< 1, b >$ (where $< 1, b >$ is the real subgroup generated by 1 and b). So if $(a)_b$ is integer, then the real numbers a, b and 1 are linearly dependent on $\mathbb{Z}$ and $\mathbb{Q}$. The converse is not valid (the conditions are not sufficient), because if $b = \sqrt{2}$ and $a = 2\sqrt{2} + 2$, then $a \in< 1, b >$ and a, b and 1 are linearly dependent, and $(a)_b = 2 - \sqrt{2}$. But the necessary and sufficient condition for $(a)_b$ to be an integer is that a belongs to a subset of $< 1, b >$ as following:

$$\{m + kb | k \in \mathbb{Z}, m \in \mathbb{Z} \cap \mathbb{R}_b\},$$

because in this case

$$(a)_b = (m + kb)_b = (m)_b = b(\frac{m}{b})_1 = b\frac{m}{b} = m.$$

(its converse is clear). Also in general we have the following inferences:

$$a, b \in \mathbb{Q} \Rightarrow (a)_b \in \mathbb{Q}\ , \quad a \in \mathbb{Q}^c \ \&\ b \in \mathbb{Q} \Rightarrow (a)_b \in \mathbb{Q}^c$$

$$a \in \mathbb{Q} \setminus \mathbb{R}_b \ \&\ b \in \mathbb{Q}^c \Rightarrow (a)_b \in \mathbb{Q}^c.$$

In the case a and b are irrationals, if the real numbers a, b and 1 are linearly independent, then $(a)_b$ is also irrational.
Now we prove a necessary and sufficient conditions for the remainder of the generalized division of a by b to be integer number.

Lemma 1.2. If $b \neq 0$ is a rational number, then $(a)_b$ is integer if and only if a and b have the reduced rational forms $a = \frac{\alpha}{\beta}$ and $b = \frac{\gamma}{\lambda}$ (i.e. $\beta, \lambda \in \mathbb{Z}^+$ and $\gcd(\alpha, \beta) = \gcd(\gamma, \lambda) = 1$)

such that

$$\beta|\gcd(\lambda,(\alpha)_\gamma) \quad , \quad (\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}.$$

Proof. If $b \in \mathbb{Q}$ and $(a)_b \in \mathbb{Z}$, then $a \in \mathbb{Q}$, clearly. So there exist integers α, γ and positive integers β, λ for which $\gcd(\alpha,\beta) = \gcd(\gamma,\lambda) = 1$ and $a = \frac{\alpha}{\beta}$, $b = \frac{\gamma}{\lambda}$. Now putting $\theta = [\frac{a}{b}]$ we have $(a)_b = \frac{\alpha\lambda-\beta\theta\gamma}{\beta\lambda}$ thus $\beta\lambda|\alpha\lambda - \beta\theta\gamma$ and so $\beta|\lambda$, $\lambda|\beta\theta$. Therefore there exists integer d such that $[\frac{a}{b}] = [\frac{\alpha\lambda}{\beta\gamma}] = \theta = \frac{\lambda}{\beta}d$ and this implies $\frac{\alpha}{\gamma} - \frac{\beta}{\lambda} < d \leq \frac{\alpha}{\gamma}$. But since $\frac{\beta}{\lambda} \leq 1$, then

$$\frac{\alpha}{\gamma} - \frac{\beta}{\lambda} < d = [\frac{\alpha}{\gamma}] = \frac{\alpha}{\gamma} - (\frac{\alpha}{\gamma}).$$

So $(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ and

$$(a)_b = \frac{\alpha\lambda - \beta\theta\gamma}{\beta\lambda} = \frac{\alpha - d\gamma}{\beta} = \frac{(\alpha)_\gamma}{\beta},$$

therefore $\beta|\gcd(\lambda,(\alpha)_\gamma)$.
Conversely suppose that the conditions are held. Then $\beta|\lambda$ and $(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ imply $[\frac{a}{b}] = [\frac{\lambda}{\beta}\frac{\alpha}{\gamma}] = \frac{\lambda}{\beta}[\frac{\alpha}{\gamma}]$ (considering the next note) and so $(a)_b = \frac{\alpha}{\beta} - \frac{\gamma}{\lambda}\frac{\lambda}{\beta}[\frac{\alpha}{\gamma}] = \frac{(\alpha)_\gamma}{\beta} \in \mathbb{Z}$. □

Note: For every real numbers x and $\kappa \neq 0$ we have

$$[\kappa x] = \kappa[x] \Leftrightarrow (\kappa x) = \kappa(x) \Leftrightarrow (x) = (x)_{\frac{1}{\kappa}} \Rightarrow (x) < |\frac{1}{\kappa}|,$$

and the converse of the last conclusion is valid if $\kappa = k$ is a natural number ($x \in [0, \frac{1}{k}) + \mathbb{Z} \Leftrightarrow (x) < \frac{1}{k} \Leftrightarrow (x) = (x)_{\frac{1}{k}}$). So we conclude that the condition $(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ in the above theorem can be replaced by $[\frac{\lambda}{\beta}\frac{\alpha}{\gamma}] = \frac{\lambda}{\beta}[\frac{\alpha}{\gamma}]$.

Corollary 1.3. Let a, b be reduced rational numbers $a = \frac{\alpha}{\beta}$ and $b = \frac{\gamma}{\lambda}$.
(i) A necessary condition on a and b for $(a)_b$ to be an integer is

$$\lambda(\frac{\alpha}{\gamma})_1 < \beta \leq \min\{\lambda, |(\alpha)_\gamma|\}.$$

Hence if $b \in \mathbb{Z}$ then we should have $a \in \mathbb{Z}$. Also (in that case) if $\beta \nmid \lambda$ or $\beta \nmid (\alpha)_\gamma$ or $\beta \geq |\gamma|$ or $\beta \leq \lambda(\frac{\alpha}{\gamma})_1$, then $(a)_b$ is a non-integer rational number.
(ii) If $b > 0$, then the b-bounded remainder of the (generalized) division of a by b is an integer if and only if $\beta|\gcd(\lambda, \text{the remainder of the division of } \alpha \text{ by } \gamma\ )$ and $(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ (notice that the identity $\frac{a}{b} = \frac{\alpha\lambda}{\beta\gamma}$ implies there exists another remainder for the division a by b for which is $\beta\gamma$-bounded and can be gotten from the ordinary division algorithm).

2 Finite b-Representation of Real Numbers

In [3] some applications of b-parts for the infinite digital b-expansion of real numbers (to the base integer $b \neq 0, \pm 1$) were studied. Also some direct formula for their digits (using b-parts) were stated. The followings are their summary.

We call a function $a : \mathbb{Z} \to S$ (where $S \neq \emptyset$ is an arbitrary set) a "two sided sequence " and denote it by $\{a_n\}_{+\infty}^{-\infty}$.

Definition 2.1. Let $b > 1$ be a fixed positive integer. A *b-digital sequence* (to base b) is a two-sided sequence $\{a_n\}_{+\infty}^{-\infty}$ of integers which satisfy the following conditions
i) $0 \leq a_n < b \ : \ \forall n \in \mathbb{Z}$,
ii) there exists an integer N such that $a_n = 0$, for all $n > N$
iii) for every integer m, there exists an integer $n \leq m$ such that $a_n \neq b - 1$.
In fact N is the largest integer that $a_N \neq 0$ (we set $N = 0$, for the zero b-digital sequence).

Theorem 2.2(Fundamental theorem of b-digital sequences). Let $b > 1$ be a positive integer. A two-sided sequence $\{a_n\}_{+\infty}^{-\infty}$ of integers is a b-digital sequence if and only if there exists a nonnegative real a such that

$$a_n = ([b^{-n}a])_b \quad : \forall n \in \mathbb{Z}.$$

More over in this case we have:

$$a_n = ([ab^{-n}])_b = [(ab^{-n})_b] = (ab^{-n})_b - (ab^{-n}) = (ab^{-n})_b - ((ab^{-n})_b)$$

$$= [ab^{-n}] - [[ab^{-n}]]_b = [ab^{-n}] - [ab^{-n}]_b,$$

for all $n \in \mathbb{Z}$. Also the number N (that is described in the above definition and $N + 1$ is the number of its integer part's digits) is equal to $[\log_b a]$.

Proof. See [3], for a proof by using b-parts.

Theorem 2.3 Fix an integer $b \neq 0, \pm 1$ and a real number $a \neq 0$ and put $\delta_n = \text{sgn}(ab^n)$, where sgn is the signum function. There is a unique two-sided sequence of integers a_n such that

$$a = \sum_{+\infty}^{-\infty} a_n b^n,$$

where a_n satisfy the following conditions
i) $|a_n| < |b| \quad : \ \forall n \in \mathbb{Z}$,
ii)$a_n = 0$ or $\text{sgn}(a_n) = \delta_n$, for all n,
iii) For every m there exits $n \leq m$ such that $a_n \neq \delta_n(|b| - 1)$.
Moreover we have

$$a_n = \delta_n([|b|^{-n}|a|])_{|b|} \quad : \forall n \in \mathbb{Z}.$$

Proof. See [3].

The generalized division algorithm induces this idea that perhaps we can generalize the base $b \neq 0, \pm 1$, from integers to all $b \in \mathbb{R} \setminus \{0, \pm 1\}$. In this case the method is different and the representation is finite and unique but is not digital. If a, b are positive integers, then it reduces to the ordinary representation a to the base b. In fact we will prove a necessary and sufficient conditions for the finite b-representation to be digital. Of course one can see several different representations for real numbers. For example, there is an infinite digital representation to the base $q \in [1, 2)$ with coefficients $0, 1$, that it is not unique (necessarily) and is not usable for all positive real numbers (see [2]).
Now let start it by an important theorem.

Theorem 2.4. Fix real $b > 1$. For any real $a \geq b$ $[0 < a < b]$ there exists a unique positive integer [nonnegative integer] N and a unique finite real sequence $\{a_n\}_0^N$ such that
i) $a = \sum_0^N a_n b^n$,
ii) $a_n \in [0, b)$: for all $0 \leq n \leq N$,
iii) $a_n = q_n - bq_{n+1}$: for all $0 \leq n \leq N$,
where $q_1, \cdots, q_N$ are positive integers and $q_{N+1} = 0$.

(We call the finite sequences $\{a_n\}_0^N$, $\{q_n\}_0^{N+1}$ *finite b-bounded sequence of* a and *finite b-quotient sequence of* a to the base b, respectively).

Proof. Let $a \geq b$. Considering the generalized division algorithm, there exist $r \in [0, b)$, $q \in \mathbb{N}(\mathbb{N}^* = \mathbb{Z}^+, \mathbb{N} = \mathbb{N}^* \cup \{0\})$ such that $a = bq + r$ $(q = [\frac{a}{b}] \geq 1)$. Set $q_0 = a$, $q_1 = q$ and $a_0 = r$. If $q_1 < b$, then putting $N = 1$, $a_1 = q_1$ and $q_2 = 0$ the conditions hold. Now if $q_1 \geq b$, then we construct the sequences $\{a_n\}$, $\{q_n\}$ as follows.
Suppose a_n and q_{n+1} have been constructed (for $n \geq 0$). Applying the generalized division algorithm, there exist $0 \leq a_{n+1} < b$ and $q_{n+2} \in \mathbb{N}$ such that $a_{n+1} = q_{n+1} - bq_{n+2}$ so

$$q_{n+2} = [\frac{q_{n+1}}{b}] \leq \frac{q_{n+1}}{b} < q_{n+1},$$

(because $b > 1$, $q_{n+1} \in \mathbb{N}^*$). Since $q_1 > q_2 > \cdots$ and these are nonnegative integers, then there exists the least positive integer N such that $q_N \neq 0$ and $q_{N+1} = 0$. Therefore the finite sequences $\{a_n\}_0^N$, $\{q_n\}_0^{N+1}$ have been produced such that $0 \leq a_n < b$ and $a_n = q_n - bq_{n+1}$ for all $0 \leq n \leq N$ so

$$\sum_0^N a_n b^n = \sum_0^N (q_n b^n - q_{n+1} b^{n+1}) = q_0 - q_{N+1} b^{N+1} = a.$$

Now assume that sequences $\{a_n\}_0^N$, $\{q_n\}_0^{N+1}$ satisfy the conditions, then $a = q_0 - q_{N+1} b^{N+1} = q_0$. Also

$$a_n = (a_n)_b = (q_n - bq_{n+1})_b = (q_n)_b \quad : \; n = 0, \cdots, N$$

so, $q_{n+1} = [b^{-1}q_n]_1$,therefore,

$$q_{n+1} = [b^{-1}q_n] \ \ : \text{ for all } 0 \le n \le N, \ q_0 = a$$

$$(1) \quad a_{n+1} = (q_{n+1})_b = ([b^{-1}q_n])_b \ \ : \text{ for all } 0 \le n \le N-1, \ a_0 = (a)_b.$$

<u>Uniqueness</u>: Let the couple sequences $\{a_n\}_0^N$, $\{q_n\}_0^{N+1}$ and $\{a_n^{'}\}_0^{N^{'}}$, $\{q_n^{'}\}_0^{N^{'}+1}$ satisfy the conditions. If $N < N^{'}$, then the relation (1) implies that $q_n = q_n^{'}$, $a_n = a_n^{'}$ for all $0 \le n \le N$. So

$$q_N^{'} = q_N = a_N = a_N^{'} = q_N^{'} - bq_{N+1}^{'},$$

therefore $q_{N+1}^{'} = 0$ so $0 = q_{N+1}^{'} = \cdots = q_{N^{'}}^{'}$ and so $0 = a_{N+1}^{'} = \cdots = a_{N^{'}}^{'}$, but this is a contradiction (because $a_{N^{'}}^{'}$ is not zero). Similarly $N^{'} \not< N$. Therefore $N = N^{'}$ and the first part of the proof is complete. Now if $0 < a < b$, then putting $N = 0$, $q_0 = a_0 = a = (a)_b$ and $q_1 = 0$ the conditions (i), (ii), (iii) are hold. For uniqueness, if there exists $N \ge 1$ and a finite sequence $\{a_n\}_0^N$ such that the conditions are hold, then

$$a_N = q_N - bq_{N+1} = q_N \ge 1.$$

So $a = \sum_0^N a_n b^n \ge a_N b^N \ge b^N \ge b$ thus $a \ge b$ and this is a contradiction. □

Note. In the above theorem always $q_0 = a$, $a_0 = (a)_b$ and if $a \ge b$, then a_N always is a natural number. For N we have

$$N = 0 \Leftrightarrow a < b \ \ , \ \ N = 1 \Leftrightarrow b \le a < b^2 + (a)_b \ \ , \ \ N > 1 \Leftrightarrow a \ge b^2 + (a)_b.$$

In case $a = 0$ we set $N = 0$ (and $a_0 = (0)_b = 0$). Now if $N > 1$, then

$$a_{N-1} \in \mathbb{Q} \Leftrightarrow b \in \mathbb{Q} \Leftrightarrow a_1, a_2 \cdots a_N \in \mathbb{Q} \Leftrightarrow a_{n_0} \in \mathbb{Q} \ \text{ for some } 1 \le n_0 \le N-1,$$

$$a_{N-1}, a_0 \in \mathbb{Q} \Leftrightarrow a, b \in \mathbb{Q},$$

$$a_{N-1} \in \mathbb{Q}^c \Leftrightarrow b \in \mathbb{Q}^c \Leftrightarrow a_1, a_2 \cdots a_{N-1} \in \mathbb{Q}^c \Leftrightarrow a_{n_0} \in \mathbb{Q}^c \ \text{ for some } 1 \le n_0 \le N.$$

Also if $b \in \mathbb{Q}^c$, then the condition $q_{N+1} = 0$ in the theorem can be replaced by $q_{N+1} \in \mathbb{Q}$, $a_N \in \mathbb{N}$.

Theorem 2.5 (Unique finite b-representation of real numbers). Fix real number $b \ne 0, \pm 1$ and put $\varepsilon = \text{sgn}(|b| - 1)$. For every real a there exists a unique nonnegative integer N and a finite real sequence $\{a_n\}_0^N$ such that

i) $a = \sum_0^N a_n b^{\varepsilon n}$,

ii) $a_n \in \delta_n[0, |b|^{\varepsilon})$: for all $0 \le n \le N$,

where $\delta_n = \text{sgn}(ab^n)$

iii) $a_n = q_n - b^{\varepsilon} q_{n+1}$: for all $0 \le n \le N$,

where $q_n \in \delta_n \mathbb{N}^*$, for all $1 \le n \le N$, and $q_{N+1} = 0$.

(Notice that $|\varepsilon| = |\delta_n| = 1$, $b^\varepsilon = b^{\pm 1}$ and we have $\delta_n[0, |b|^\varepsilon) = \delta_{n+1}[0, b^\varepsilon)$ or $\delta_{n+1}(b^\varepsilon, 0]$, for comparing this theorem and Theorem 2.4.)

Proof. Put $\alpha = |a|$, $\beta = |b|^\varepsilon$. Since $\beta > 1$, then Theorem 2.4 implies that there exists a nonnegative integer N ($N = 0$ if and only if $0 \leq \alpha < \beta$) and finite positive real sequence $\{\alpha_n\}_0^N$ such that

$$\alpha = \sum_0^N \alpha_n \beta^n \Rightarrow a = \sum_0^N \mathrm{sgn}(a)\mathrm{sgn}(b^{\varepsilon n})\alpha_n b^{\varepsilon n}$$

Putting $\delta_n = \mathrm{sgn}(ab^n) = \mathrm{sgn}(ab^{\varepsilon n})$ and $a_n = \delta_n \alpha_n$ we have $a = \sum_0^N a_n b^{\varepsilon n}$ and $a_n = \delta_n \alpha_n \in \delta_n[0, \beta) = \delta_n[0, |b|^\varepsilon)$. But we have

$$\delta_n |b|^\varepsilon = \mathrm{sgn}(a)\mathrm{sgn}(b^{\varepsilon n})\mathrm{sgn}(b^\varepsilon) b^\varepsilon = \delta_{n+1} b^\varepsilon,$$

Therefore

$$\delta_n[0, |b|^\varepsilon) = \delta_{n+1}[0, 1)b^\varepsilon = \delta_{n+1}[0, b^\varepsilon) \text{ or } \delta_{n+1}(b^\varepsilon, 0].$$

On the other hand $\alpha_n = Q_n - \beta Q_{n+1}$ for all $0 \leq n \leq N$ where $Q_1, \cdots, Q_n$ are positive integers and $Q_{N+1} = 0$. So putting $q_n = \delta_n Q_n$ and considering the above relation, we have $q_n \in \delta_n \mathbb{N}^*$ and $q_{N+1} = 0$ and

$$a_n = \delta_n Q_n - \delta_n Q_{n+1} |b|^\varepsilon = q_n - q_{n+1} b^\varepsilon.$$

Note that N,$\{a_n\}_0^N$ are unique, considering the above relations and Theorem 2.4. □

Definition 2.6. Fix the real number $b \neq 0, \pm 1$. For all real a we call the finite b-bounded sequence $\{a_n\}_0^N$ to the base b, the (generalized) finite representation a to the base b and write

$$(2) \quad a = < a_N >< a_{N-1} > \cdots < a_0 >_b .$$

In this representation we call every a_n *b-parcel* of a and denote it by $\mathrm{dgt}^*_{n,b}(a)$ or $\mathrm{prl}_{n,b}(a)$.

Notice that we use the notation $\mathrm{dgt}_{n,b}(a)$, only for the case that the expansion is digital (a_n are integers for all n). If a, b are natural numbers, then Theorem 2.4 reduces to the b place value notation for a and the symbols $<>$ can be removed in the representation (2), i.e. $a = a_N a_{N-1} \cdots a_{0_b}$

Example 2.7. The following is a unique finite digital $\frac{41}{4}$-representation:

$$\frac{992653}{2} = < 4 >< 3 >< 9 >< 9 >< 1 >< 1 >_{\frac{41}{4}} .$$

Lemma 2.8. Consider the number N in the finite b-representation of a (that $N+1$ is the number of the b-parcels of a).
If $a \geq b > 0$, then $N \leq [\log_b a]$. In general $N \neq [\log_b a]$, but if b is a positive integer, then $N = [\log_b a]$ and $q_n = [b^{-n}a]$, $a_n = ([b^{-n}a])_b$, for all $n \geq 1$ (but not for $n = 0$).

Proof. If $k = [\log_b a]$, then $0 \leq ab^{-k-1} < 1$, on the other hand (1) implies that $q_{k+1} \leq ab^{-k-1}$ so $q_{k+1} = 0$ hence $N \leq k$. In general $N \neq [\log_b a]$ for if $a = \pi$, $b = \sqrt{2}$, then

$$\pi =< 1 >< 2 - \sqrt{2} >< \pi - 2\sqrt{2} >_{\sqrt{2}}$$

so $N = 2 \neq [\log_{\sqrt{2}} \pi]$. But if b is a positive integer, then $q_n = [b^{-n}a]$ and $a_n = ([b^{-n}a])_b$, for all $n \geq 1$, considering (1) and the property (V). So $N = [\log_b a]$, considering Theorem 2.2. $\square$

Remark 2.9. In general if $a \geq b > 1$, then

$$\mathrm{dgt}^*_{n,b}(a) = ([b^{-1}[b^{-1} \cdots [b^{-1}a] \cdots]])_b \ \ (n \text{ times}) \ \ : \forall n \geq 1,$$

(by (1)) and $\mathrm{dgt}^*_{0,b}(a) = (a)_b$ (for all $a \geq 0$) and we have

$$a = (a)_b + ([b^{-1}a])_b + ([b^{-1}[b^{-1}a]])_b + \cdots = \sum_{n=0}^{\infty} \mathrm{dgt}^*_{n,b}(a) b^n$$

$$= \sum_{n=0}^{[\log_b a]} \mathrm{dgt}^*_{n,b}(a) b^n = \sum_{n=0}^{N} \mathrm{dgt}^*_{n,b}(a) b^n,$$

for all $a \geq 1$, $b > 1$ (note that in the above series $\mathrm{dgt}^*_{n,b}(a) = 0$, for all $n > N$).
If $b \in \mathbb{N}$, then

$$\mathrm{dgt}^*_{n,b}(a) = \mathrm{dgt}_{n,b}(a) = ([b^{-n}a])_b \ \ : \ \forall n \geq 1,$$

but for $n = 0$ we have $\mathrm{dgt}_{0,b}(a) = ([a])_b$, $\mathrm{dgt}^*_{0,b} = (a)_b$ and

$$\mathrm{dgt}^*_{0,b}(a) = (a)_b = ([a])_b + (a) = \mathrm{dgt}_{0,b}([a]) + \sum_{-1}^{-\infty} \mathrm{dgt}_{n,b}(a) b^n,$$

($\mathrm{dgt}^*_{n,b}(a)$ is not defined for $n < 0$).

Now we prove the necessary and sufficient conditions for the finite b-represetation to be digital.

Theorem 2.10. Let $a > b > 0$ be real numbers. The finite representation of a to the base b is digital (b-parcels are b-digits) if and only if a and b have the reduced rational forms $a = \frac{\alpha}{\beta}$ and $b = \frac{\gamma}{\lambda}$ such that

$$(3) \quad \beta | \gcd(\lambda, (\alpha)_\gamma) \quad , \quad \max\{\frac{1}{\beta}(\frac{\alpha}{\gamma})_1, (\frac{q_1}{\gamma})_1, (\frac{q_2}{\gamma})_1, \cdots, (\frac{q_N}{\gamma})_1\} < \frac{1}{\lambda},$$

where $q_1 = [\frac{a}{b}]$ and $q_{n+1} = [\frac{q_n}{b}]$, for $n \geq 1$.

Proof. If the representation is digital, then $b \in \mathbb{Q}$, considering $N > 1$ (because $a > b$) and

the condition (iii) of the representation. Moreover if $a_0 \in \mathbb{Q}$, then $a \in \mathbb{Q}$ and Lemma 1.2 implies a and b have the reduced rational forms $a = \frac{\alpha}{\beta}$ and $b = \frac{\gamma}{\lambda}$ such that $\beta | \gcd(\lambda, (\alpha)_\gamma)$ and $\frac{1}{\beta}(\frac{\alpha}{\gamma})_1 < \frac{1}{\lambda}$. Also the condition (iii) of Theorem 2.4 implies $\lambda | q_{n+1} = [\lambda \frac{q_n}{\gamma}]$, for every natural number n. Now we get (3), considering the following relations (4) and (5) :
Notice that if $\kappa \geq 1$ is a real number, then

$$(4) \quad [x]_{\frac{1}{\kappa}} \in \mathbb{Z} \Leftrightarrow [\kappa x] \in \kappa\mathbb{Z} \Leftrightarrow [\kappa x] = \kappa[x] \Leftrightarrow (\kappa x) = \kappa(x)$$

$$\Leftrightarrow (x) = (x)_{\frac{1}{\kappa}} \Rightarrow (x) < |\frac{1}{\kappa}|,$$

and so if $\kappa = k$ is a natural number, then

$$(5) \quad [x]_{\frac{1}{k}} \in \mathbb{Z} \Leftrightarrow k|[kx] \Leftrightarrow [kx] = k[x] \Leftrightarrow (x) < \frac{1}{k} \Leftrightarrow x \in [0, \frac{1}{k}) + \mathbb{Z} \Leftrightarrow (x) = (x)_{\frac{1}{k}}.$$

Conversely, if (3) is held, then Lemma1.2, the condition (iii) of the representation and (5) imply that the representation is digital. □

Example. The followings are some digital finite b-representations which come from Theorem 2.4 and it can be seen that the conditions of the above theorem hold.

$$\frac{9}{2} =< 1 >< 2 >_{\frac{5}{2}} \ , \ 16 =< 3 >< 3 >_{\frac{13}{3}}$$

$$100 =< 6 >< 11 >_{\frac{89}{6}} \ , \ \frac{737}{2} =< 4 >< 0 >< 0 >< 4 >_{\frac{9}{2}} \ .$$

If $0 < b < 1$, then we can have another unique finite representation of a which a_ns are decimal numbers. In this case the range values of a_n-s are $[0, 1)$ (instead $[0, b)$).

Theorem 2.11. Fix real $0 < b < 1$. For any real $a \geq \frac{1}{b}$ there exists a unique positive integer N and a unique finite real sequence $\{a_n\}_0^N$ such that
i) $a = \sum_0^N a_n b^{-n-1}$,
ii) $0 \leq a_n < 1$: for all $0 \leq n \leq N$,
iii) $a_n = bq_n - q_{n+1}$: for all $0 \leq n \leq N$,
where $q_1, \cdots, q_N$ are positive integers and $q_{N+1} = 0$

Proof. Put $\beta = \frac{1}{b}$. Since $a \geq \beta > 1$, then Theorem 2.4 implies there exist a unique positive integer N and a unique positive real sequence $\{\alpha_n\}_0^N$ such that $a = \sum_0^N \alpha_n \beta^n$. Putting $a_n = b\alpha_n$ we have $0 \leq a_n < 1$ and $a = \sum_0^N a_n b^{-n-1}$. Also $\alpha_n = q_n - \beta q_{n+1}$ implies $a_n = bq_n - q_{n+1}$. In fact considering (1) it can be seen that $a_0 = (ba)_1$, $q_0 = a$ and $a_{n+1} = (b[bq_n]_1)_1$ for all $0 \leq n \leq N-1$. Therefore N,$\{a_n\}_0^N$ are unique, considering Theorem 2.4. □

References

[1] Apostol, T.M. *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

[2] Glendinning, P., N. Sidorov, Unique Representations of Real Numbers in Non-Integer Bases, *Math. Res. Lett.*, Vol. 8, 2001, 535–543.

[3] Hooshmand, M.H. b-Digital Sequences, *Proceedings of the 9th world Multiconference on Systemics, Cybernetics and Informatics* (WMSCI 2005)- Orlando, USA, 142–146.

[4] Hooshmand, M.H., H. Kamarul Haili, Some Algebraic Properties of b-Parts of Real Numbers, *Šiauliai Math. Semin.*, Vol. 3, 2008, No. 11, 115–121.

[5] Hooshmand, M.H., H. Kamarul Haili, Decomposer and Associative Functional Equations, *Indag. Mathem.*, N.S., Vol. 18, 2007, No. 4, 539–554.