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A note on a Diophantine equation

József Sándor

Babeş-Bolyai University, Department of Mathematics
Str. Kogălniceanu nr. 1, 400084 Cluj-Napoca, Romania
e-mail: jsandor@math.ubbcluj.ro

Abstract: We offer an elementary approach to the solution of diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{2}$, considered recently in Vol. 19, No. 3 of this journal. An extension is provided, too.
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1 Introduction

In the recent paper [1] the solution of diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2} \tag{1}$$

is offered. In the proof, an auxiliary result from paper [2] has been used.

In what follows, we shall point out that, equation can be solved elementary, without the use of any auxiliary result.

2 The proof

We may assume $x \le y \le z$.

As $\frac{1}{x} < \frac{1}{2}$, we get $x \ge 3$. On the other hand, as $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge \frac{3}{z}$, by $\frac{1}{2} \ge \frac{3}{z}$ we get $z \ge 6$. Similarly, as $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le \frac{3}{x}$, we get $x \le 6$. Thus for x the following cases are possible: $x \in \{3, 4, 5, 6\}$. This leads to the following four equations:

$$x = 3, \quad \frac{1}{y}+\frac{1}{z}=\frac{1}{6}, \tag{2}$$

$$x = 4, \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{4}, \tag{3}$$

$$x = 5, \quad \frac{1}{y} + \frac{1}{z} = \frac{3}{10}, \tag{4}$$

$$x = 6, \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{3}. \tag{5}$$

Remark that equations (2), (3), (5) may be rewritten as

$$(y-6)(z-6) = 36, \tag{2'}$$

$$(y-4)(z-4) = 16, \tag{3'}$$

$$(y-3)(z-3) = 9. \tag{5'}$$

As $z \geq 6$ in $(2')$ and $y - 6 \leq z - 6$, for $(2')$ only the following cases are possible:

$$y-6 = 1,\ z-6 = 36;\ y-6 = 2,\ z-6 = 18;\ y-6 = 3,\ z-6 = 12;$$

$$y-6 = 4,\ z-6 = 9;\ y-6 = 6,\ z-6 = 6$$

leading to the solutions

$$(x, y, z) = (3, 7, 42);\ (3, 8, 24);\ (3, 9, 18);\ (3, 10, 15);\ (3, 12, 12).$$

In a same manner, equation $(3')$ leads to

$$(x, y, z) = (4, 5, 20);\ (4, 6, 12);\ (4, 8, 8),$$

while $(5')$ to

$$(x, y, z) = (6, 6, 6).$$

Equation (4) gives by $\frac{1}{y} + \frac{1}{z} \leq \frac{2}{y}$ that is $\frac{3}{10} \leq \frac{2}{y}$, so $y \leq 6$. Since $y \geq x = 5$, we have two cases: $y = 5$ and $y = 6$. There is solution only for $y = 5$, giving:

$$(x, y, z) = (5, 5, 10).$$

Remark. We should note that in Theorem 2.3 of [1], the set of solutions (x, y, z) with $x \leq y \leq z$ is provided. Clearly any permutation of (x, y, z) is a solution, too.

3 An extension

A more general equation than (1) is

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{p}{q}, \tag{6}$$

where $p, q \geq 1$ are given positive integers.

The above method can be extended in order to prove that equation (6) has a finite number of solutions, which can be determined in theory.

Indeed, let us suppose again $x \leq y \leq z$. Then $\frac{3}{z} \leq \frac{p}{q} \leq \frac{3}{x}$ implies

$$x \leq \frac{3q}{p} \leq z, \tag{7}$$

where $x > \frac{q}{p}$, as $\frac{1}{x} < \frac{p}{q}$. This shows that the possible values of x lie between $\left[\frac{q}{p}\right] + 1$ and $\left[\frac{3q}{p}\right]$; i.e. a finite number of values. Let $x = a$ be such a value. Then from (6) we get

$$\frac{1}{y} + \frac{1}{z} = \frac{p'}{q'}, \tag{8}$$

where $\frac{p'}{q'} = \frac{p}{q} = \frac{1}{a}$. Again, as $\frac{p'}{q'} \leq \frac{2}{y}$, we get $a \leq y \leq \frac{2q'}{p'}$, so a finite number of values. Finally, for $y = b$, with $a \leq b \leq \frac{2q'}{p'}$ one obtains

$$\frac{1}{b} + \frac{1}{z} = \frac{p'}{q'}, \tag{9}$$

with possible solutions $z = bq'/(p'b - q')$, in case if this is an integer. Therefore the number of values of z is finite, too.

References

[1] Rabago, J.F.T., R.P. Tagle, On the area and volume of a certain regular solid and the diophantine equation $\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, *Notes Numb. Th. Discr. Math.*, Vol. 19, 2013, No. 3, 28–37.

[2] Zelator, K. An ancient Egyptian problem: The diophantine equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ (preprint).