A note on a Diophantine equation

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Abstract: We offer an elementary approach to the solution of diophantine equation \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2} \), considered recently in Vol. 19, No. 3 of this journal. An extension is provided, too.

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1 Introduction

In the recent paper [1] the solution of diophantine equation

\[ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2} \]  

is offered. In the proof, an auxiliary result from paper [2] has been used.

In what follows, we shall point out that, equation can be solved elementary, without the use of any auxiliary result.

2 The proof

We may assume \( x \leq y \leq z \).

As \( \frac{1}{x} < \frac{1}{2} \), we get \( x \geq 3 \). On the other hand, as \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{3}{z} \), by \( \frac{1}{2} \geq \frac{3}{z} \) we get \( z \geq 6 \).

Similarly, as \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{x} \), we get \( x \leq 6 \). Thus for \( x \) the following cases are possible: \( x \in \{3, 4, 5, 6\} \). This leads to the following four equations:

\[ x = 3, \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{6}, \]  

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\[ x = 4, \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{4}, \quad (3) \]
\[ x = 5, \quad \frac{1}{y} + \frac{1}{z} = \frac{3}{10}, \quad (4) \]
\[ x = 6, \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{3}. \quad (5) \]

Remark that equations (2), (3), (5) may be rewritten as
\[ (y - 6)(z - 6) = 36, \quad (2') \]
\[ (y - 4)(z - 4) = 16, \quad (3') \]
\[ (y - 3)(z - 3) = 9. \quad (5') \]

As \( z \geq 6 \) in (2') and \( y - 6 \leq z - 6 \), for (2') only the following cases are possible:
\[ \begin{align*}
  y - 6 &= 1, \quad z - 6 = 36; \quad y - 6 = 2, \quad z - 6 = 18; \quad y - 6 = 3, \quad z - 6 = 12; \\
  y - 6 &= 4, \quad z - 6 = 9; \quad y - 6 = 6, \quad z - 6 = 6
\end{align*} \]
leading to the solutions
\[ (x, y, z) = (3, 7, 42); \ (3, 8, 24); \ (3, 9, 18); \ (3, 10, 15); \ (3, 12, 12). \]

In a same manner, equation (3') leads to
\[ (x, y, z) = (4, 5, 20); \ (4, 6, 12); \ (4, 8, 8), \]
while (5') to
\[ (x, y, z) = (6, 6, 6). \]

Equation (4) gives by \( \frac{1}{y} + \frac{1}{z} \leq \frac{2}{y} \) that is \( \frac{3}{10} \leq \frac{2}{y} \), so \( y \leq 6 \). Since \( y \geq x = 5 \), we have two cases: \( y = 5 \) and \( y = 6 \). There is solution only for \( y = 5 \), giving:
\[ (x, y, z) = (5, 5, 10). \]

Remark. We should note that in Theorem 2.3 of [1], the set of solutions \((x, y, z)\) with \( x \leq y \leq z \) is provided. Clearly any permutation of \((x, y, z)\) is a solution, too.

3 An extension

A more general equation than (1) is
\[ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{p}{q}, \quad (6) \]
where \( p, q \geq 1 \) are given positive integers.

The above method can be extended in order to prove that equation (6) has a finite number of solutions, which can be determined in theory.

Indeed, let us suppose again \( x \leq y \leq z \). Then \( \frac{3}{z} \leq \frac{p}{q} \leq \frac{3}{x} \) implies

\[
x \leq \frac{3q}{p} \leq z,
\]

(7)

where \( x > \frac{q}{p} \), as \( \frac{1}{x} < \frac{q}{p} \). This shows that the possible values of \( x \) lie between \( \left\lfloor \frac{q}{p} \right\rfloor + 1 \) and \( \left\lfloor \frac{3q}{p} \right\rfloor \); i.e. a finite number of values. Let \( x = a \) be such a value. Then from (6) we get

\[
\frac{1}{y} + \frac{1}{z} = \frac{p'}{q'},
\]

(8)

where \( \frac{p'}{q'} = \frac{p}{q} = \frac{1}{a} \). Again, as \( \frac{p'}{q'} \leq \frac{2}{y} \), we get \( a \leq y \leq \frac{2q'}{p'} \), so a finite number of values. Finally, for \( y = b \), with \( a \leq b \leq \frac{2q'}{p'} \) one obtains

\[
\frac{1}{b} + \frac{1}{z} = \frac{p'}{q'},
\]

(9)

with possible solutions \( z = bq'/(p'b - q') \), in case if this is an integer. Therefore the number of values of \( z \) is finite, too.

References

[1] Rabago, J.F.T., R.P. Tagle, On the area and volume of a certain regular solid and the diophantine equation \( \frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \), *Notes Numb. Th. Discr. Math.*, Vol. 19, 2013, No. 3, 28–37.

[2] Zelator, K. An ancient Egyptian problem: The diophantine equation \( \frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \) (preprint).