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A characterization of canonically consistent total signed graphs

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Abstract: The *canonical marking* on a *signed graph* (or *sigraph*, in short) S is defined as: for each vertex $v \in V(S)$, $\mu_\sigma(v) = \prod_{e_j \in E_v} \sigma(e_j)$, where E_v is the set of edges e_j incident at v in S. If S is canonically marked, then a cycle Z in S is said to be *canonically consistent* ($\mathcal{C}$*-consistent*) if it contains an even number of negative vertices and the given sigraph S is $\mathcal{C}$*-consistent* if every cycle in it is $\mathcal{C}$-consistent. The *total sigraph* $T(S)$ of a sigraph $S = (V,\ E,\ \sigma)$ has $T(S^u)$ as its underlying graph and for any edge uv of $T(S^u)$,

$$\sigma_T(uv) = \begin{cases} \sigma(uv) & \text{if } u, v \in V, \\ \sigma(u)\sigma(v) & \text{if } u, v \in E, \\ \sigma(u) \prod_{e_j \in E_v} \sigma(e_j) & \text{if } u \in E \text{ and } v \in V. \end{cases}$$

In this paper, we establish a characterization of canonically consistent total sigraphs.
Keywords: Sigraph, Canonical marking, Consistent sigraph, Total sigraph.
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1 Introduction

For standard terminology and notation in graph theory we refer to Harary [20] and West [33] and Zaslavsky [34, 35] for sigraphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges. A *signed graph* (or *sigraph* in short; see [16, 19]) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$, called the *signature* of S. Alternatively, the sigraph can be written as $S = (V, E, \sigma)$, with V, E and σ in the above sense. Let $E^+(S) = \{e \in E(G) : \sigma(e) = +\}$ and $E^-(S) = \{e \in E(G) : \sigma(e) = -\}$. The elements of $E^+(S)$ and $E^-(S)$ are called *positive* and *negative edges* of S, respectively. A sigraph is said to be *homogeneous* if all its edges are of the same sign and *heterogeneous* otherwise.

A sigraph S is called *signed-regular* if the number of positive edges, $d^+(v)$ incident at a vertex v in S, is independent of the choice of v in S and the number of negative edges, $d^-(v)$ incident at a vertex v in S is also independent of the choice of v in S, *i.e.* S is (i, j)-signed-regular, where $i = d^+(v)$ is the positive degree of v in S and $j = d^-(v)$ is the negative degree of v in S. The *edge degree* $d_e(e_j)$ of an edge e_j in a sigraph S is the total number of edges adjacent to e_j in S. The *positive* (*negative*) *edge degree* $d_e^+(e_j)$ $(d_e^-(e_j))$ of an edge e_j in S is the total number of positive (negative) edges adjacent to e_j in S. A *cycle* Z in a sigraph S is an alternating sequence of distinct vertices and edges of S, beginning and ending with the same vertex, such that the two ends of every edge in the sequence are consecutive vertices of the sequence. The cycle Z is written as $Z = (v_1, v_2, ...v_n, v_1)$.

A *marked sigraph* is an ordered pair $S_\mu = (S,\ \mu)$, where $S = (S^u, \sigma)$ is a sigraph and $\mu : V(S^u) \to \{+, -\}$ is a function from the vertex set $V(S^u)$ into the set $\{+, -\}$, called a *marking* of S. A cycle Z in S_μ is said to be *consistent* if it contains an even number of negative vertices. A given sigraph S is said be *consistent* if every cycle in it is consistent [13]. The marking μ_σ defined by

$$\mu_\sigma(v) = \prod_{e_j \in E_v} \sigma(e_j),\ v \in V(S),$$

is called the *canonical marking* (or, *$\mathcal{C}$-marking* in short) of S, where E_v is the set of edges e_j incident at v in S. In any canonically marked sigraph S, a cycle Z in S is said to be *canonically consistent* (*$\mathcal{C}$-consistent*) if it contains an even number of negative vertices and the given sigraph S is *$\mathcal{C}$-consistent* if every cycle in it is $\mathcal{C}$-consistent.

The total graph $T(G)$ of a graph G is that graph whose vertex set is $V(G) \cup E(G)$, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively and in $T(G)$ two vertices are adjacent if and only if they are adjacent or incident in G. Several properties of total graphs are investigated in literature (*see* [5], [6], [7], [8], [9], [15], [17], [26]). If all the vertices of $T(G)$ have equal degree, then it is said to be a *regular total graph.* A characterization of regular total graphs was established in [11]. A characterization of total graphs was obtained in [12]. Gavril [18] has given a linear time algorithm for the recognition of the total graphs. The algorithm is based on the breadth-first search technique. We have extended this notion of total graph of a graph to the class of sigraphs in [30]. Let $S = (V,\ E,\ \sigma)$ be any sigraph. Its *total sigraph* $T(S)$ has $T(S^u)$ as its underlying graph and for any edge uv of $T(S^u)$,

$$\sigma_T(uv) = \begin{cases} \sigma(uv) & \text{if } u, v \in V, \\ \sigma(u)\sigma(v) & \text{if } u, v \in E, \\ \sigma(u)\prod_{e_j \in E_v} \sigma(e_j) & \text{if } u \in E \text{ and } v \in V. \end{cases}$$

A sigraph S and its total sigraph $T(S)$ are displayed in Fig. 1.

A characterization of total sigraphs is given in [30] and the several properties of total sigraphs are discussed in [29] and [31].

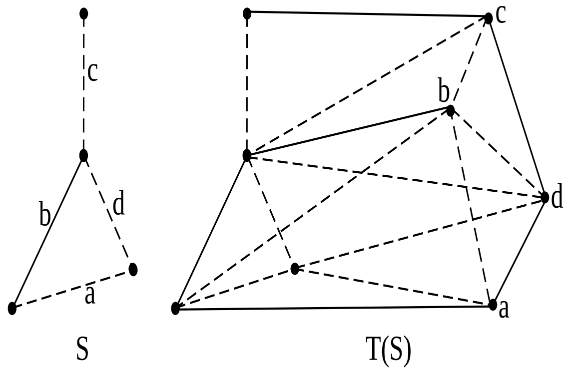

Figure 1: A sigraph S and its total sigraph $T(S)$.

2 Canonically consistent total sigraphs

Acharya and Sinha obtained consistency of sigraphs that satisfy certain sigraph equations in [4, 27]. Also, Sinha and Garg have discussed consistency of several sigraphs in [28, 29, 32]. Zaslavsky reported the following facts about μ_σ in [36]:

(i) The number of negative vertices is even [24].

(ii) The negative vertices are the odd-degree vertices of the negative subgraph, which consists of all the vertices but only the negative edges of S.

(iii) The positive edge set has no effect on μ_σ. Thus, we may assume every signed graph is a signed K_n; the positive edge set is simply the complement of the set of negative edges.

(iv) Any vertex signature $\mu : V \to \{+, -\}$ that has evenly many negative vertices is canonical with respect to some signed graph whose vertex set is V .

(v) There are a great many possible negative subgraphs that yield the same vertex signature μ_σ.

Recently there has been new interest in the canonical vertex signature in connection with deriving other sigraphs from a sigraph, in particular a line sigraph; see (in chronological order) [27, 25, 22, 36, 3]. In this note, we establish a characterization of $\mathcal{C}$-consistent total sigraphs. Towards this end, we will require the following useful result due to Hoede.

Theorem 1. [21] *A marked graph G_μ is consistent if and only if for any spanning tree T of G all fundamental cycles with respect to T are consistent and all common paths of pairs of those fundamental cycles have end vertices carrying the same marks.*

Corollary 2. *Every canonically marked signed cycle is $\mathcal{C}$-consistent.*

The *parity* of an integer states whether it is even or odd in the following theorem:

Theorem 3. *The total sigraph $T(S)$ of a sigraph $S = (S^u, \sigma)$ is $\mathcal{C}$-consistent if and only if either $S = K_2^-$ or S satisfies the following conditions:*

(i) *for each vertex $v \in V(S)$, if $d(v) \equiv 1 \pmod 2$, then $d^-(v) \equiv 0 \pmod 2$,*

(ii) *for each negative edge $e_j = uv$,*

(a) *if* $d_e^+(e_j) \equiv 0 \pmod 2$, *then* $\mu_\sigma(u) = \mu_\sigma(v)$,

(b) *if* $d_e^+(e_j) \equiv 1 \pmod 2$, *then* $\mu_\sigma(u) = -\mu_\sigma(v)$.

Proof. **Necessity**: Suppose $T(S)$ is $\mathcal{C}$-consistent. Then, every cycle in $T(S)$ is $\mathcal{C}$-consistent. If $T(S)$ is isomorphic to a heterogeneous cycle, then $S = K_2^-$. On the other hand, if $T(S)$ is not isomorphic to a heterogeneous cycle, then we shall show that the conditions (i) and (ii) hold in the sigraph S.

Suppose (i) is false. That means, for some vertex v in S, $d(v) \equiv 1 \pmod 2$ and $d^-(v) \equiv 1 \pmod 2$. It implies that the canonical marking μ_{σ_T} of $T(S)$ gives $\mu_{\sigma_T}(v) = -$. Suppose $e_i = uv$, $e_j = vw$ are two positive edges and $e_k = vx$ is a negative edge incident at v in S. Clearly, $\{e_i, e_j, e_k\} \subseteq N(v)$ in $T(S)$. Then, $Z_1 = (v, e_i, e_j, v)$ and $Z_2 = (v, e_i, e_k, v)$ are two cycles in $T(S)$. If $d^-(u) \equiv 0 \pmod 2$, then $d_T^-(e_i) \equiv 0 \pmod 2$, that means,

$$\mu_{\sigma_T}(e_i) = +.$$

Since ve_i is the common chord of the cycles Z_1 and Z_2, and v and e_i are oppositely marked in $T(S)$, it follows from Theorem 1 that at least one of the cycles Z_1, Z_2 and $Z_1 \oplus Z_2$ is $\mathcal{C}$-inconsistent in $T(S)$. This contradicts the hypothesis. Next, if $d^-(u)$ and $d^-(w)$ are both odd, then using similar argument as above,

$$\mu_{\sigma_T}(e_i) = \mu_{\sigma_T}(e_j) = +.$$

It follows that the cycle Z_1 is $\mathcal{C}$-inconsistent, a contradiction to the hypothesis.

Again, suppose $e_i = uv$, $e_j = vw$ and $e_k = vx$ are negative edges incident at v in S. If $d^+(u)$ and $d^-(u)$ are of the opposite parity, then $d_T^-(e_i) \equiv 0 \pmod 2$, that means,

$$\mu_{\sigma_T}(e_i) = +.$$

Since ve_i is the common chord of the cycles Z_1 and Z_2, and v and e_i are oppositely marked in $T(S)$, it follows from Theorem 1 that at least one of the cycles Z_1, Z_2 and $Z_1 \oplus Z_2$ is $\mathcal{C}$-inconsistent in $T(S)$. This contradicts the hypothesis. Next, if $d^+(u)$, $d^-(u)$ are of the same parity and $d^+(w)$, $d^-(w)$ are also of the same parity, then using similar argument as above,

$$\mu_{\sigma_T}(e_i) = \mu_{\sigma_T}(e_j) = -.$$

It follows that the cycle Z_1 is $\mathcal{C}$-inconsistent, a contradiction to the hypothesis. Thus, by contradiction, (i) follows.

Next, suppose $(ii)(a)$ is false. That means, for a negative edge $e_j = uv$, $d_e^+(e_j) \equiv 0 \pmod 2$ and $\mu_\sigma(u) \neq \mu_\sigma(v)$. Since $\mu_\sigma(u) \neq \mu_\sigma(v)$, $d^-(u)$ and $d^-(v)$ are of opposite parities. Without loss of generality, let $d^-(u) \equiv 1 \pmod 2$ and $d^-(v) \equiv 0 \pmod 2$. It implies that $\mu_{\sigma_T}(u) = +$, $\mu_{\sigma_T}(e_j) = -$ and $\mu_{\sigma_T}(v) = +$. Thus, we obtain a $\mathcal{C}$-inconsistent cycle $Z_3 = (u, v, e_j, u)$ in $T(S)$, a contradiction to the hypothesis. Thus, by contradiction, $(ii)(a)$ follows.

Next, suppose $(ii)(b)$ is false. That means, for a negative edge $e_j = uv$, $d_e^+(e_j) \equiv 1 \pmod 2$ and $\mu_\sigma(u) = \mu_\sigma(v)$. Since $\mu_\sigma(u) = \mu_\sigma(v)$, $d^-(u)$ and $d^-(v)$ are of the same parity. Let $d^-(u) \equiv 0 \pmod 2$ and $d^-(v) \equiv 0 \pmod 2$. It implies that $\mu_{\sigma_T}(u) = +$, $\mu_{\sigma_T}(e_j) = -$ and

$\mu_{\sigma_T}(v) = +$. Thus, we obtain a $\mathcal{C}$-inconsistent cycle $Z_4 = (u, v, e_j, u)$ in $T(S)$, a contradiction to the hypothesis. On the other hand, if $d^-(u) \equiv 1 \pmod 2$ and $d^-(v) \equiv 1 \pmod 2$ then we get $1 \pmod 2 \equiv d_e^+(e_j) = d^-(u) + d^-(v) - 2 \equiv 0 \pmod 2$, which is preposterous. Thus, $(ii)(b)$ follows by contradiction.

Sufficiency: If $S = K_2^-$, then $T(S)$ is $\mathcal{C}$-consistent. Next, suppose S satisfies conditions (i) and (ii). We shall show that $T(S)$ is a $\mathcal{C}$-consistent sigraph. Suppose $d(v) \equiv 0 \pmod 2$ for $v \in V(S)$, then $\mu_{\sigma_T}(v) = +$. Next, let $d(v) \equiv 1 \pmod 2$, then due to condition (i), $d^-(v) \equiv 0 \pmod 2$. It implies that $\mu_{\sigma_T}(v) = +$.

Further, suppose $e_j = uv$ is a positive edge in S, then $\mu_{\sigma_T}(e_j) = +$. Next, let $e_j = uv$ be a negative edge in S. If $d_e^+(e_j) \equiv 0 \pmod 2$, then using condition $(ii)(a)$, $\mu_\sigma(u) = \mu_\sigma(v)$. That means, $d^-(u)$ and $d^-(v)$ are of the same parity. This implies that $\mu_{\sigma_T}(e_j) = +$. Again, if $d_e^+(e_j) \equiv 1 \pmod 2$, then using condition $(ii)(b)$,

$$\mu_\sigma(u) = -\mu_\sigma(v).$$

That means, $d^-(u)$ and $d^-(v)$ are of opposite parities. This implies that $\mu_{\sigma_T}(e_j) = +$. Thus, each vertex in $T(S)$ is positively marked under canonical marking. Hence $T(S)$ is $\mathcal{C}$-consistent. $\square$

A sigraph S and its $\mathcal{C}$-consistent total sigraph $T(S)$ are displayed in Figure 2.

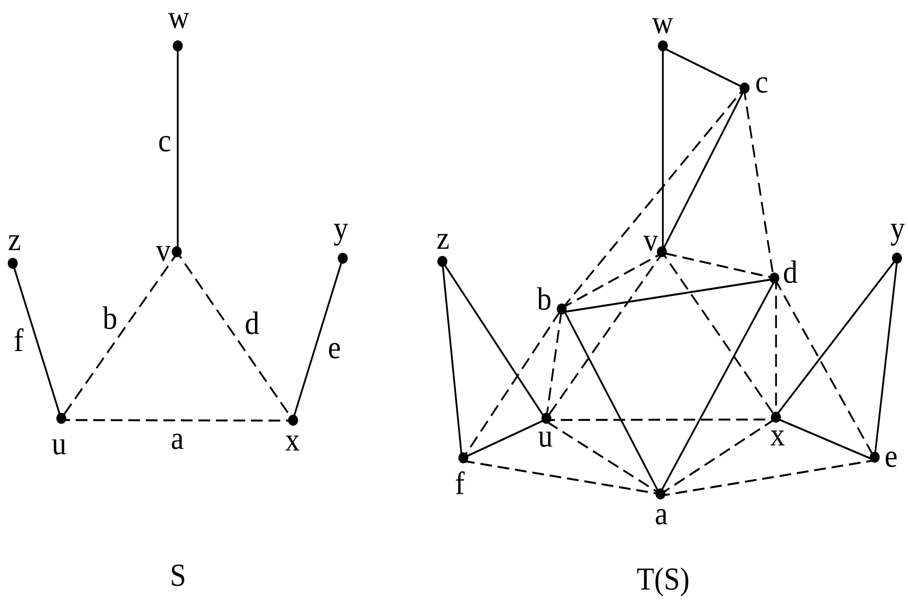

Figure 2: Showing a sigraph and its $\mathcal{C}$-consistent total sigraph.

Corollary 4. *Let S be a signed-regular sigraph. The total sigraph $T(S)$ of the sigraph S is $\mathcal{C}$-consistent if and only if either $S = K_2^-$ or if $d(v) \equiv 1 \pmod 2$ for any vertex $v \in V(S)$, then $d^-(v) \equiv 0 \pmod 2$.*

Proof. Since the given sigraph S is signed-regular, the conditions $(ii)(a)$ and $(ii)(b)$ hold for S. Now, the corollary follows due to condition (i) of Theorem 3. $\square$

Corollary 5. *Let $S = (S^u, \sigma)$ be a signed-regular sigraph. If S^u is an Eulerian graph, then the total sigraph $T(S)$ of S is $\mathcal{C}$-consistent.*

Proof. This follows from Corollary 4. $\square$

$\psi(G)$ denotes the set of all sigraphs whose underlying graph is G in the following corollary:

Corollary 6. *Let $S \in \psi(G)$, where G be a cycle. Then the total sigraph $T(S)$ is $\mathcal{C}$-consistent.*

3 Conclusion

In this paper, we have established a characterization of canonically consistent total sigraphs.

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