A note on the modified $q$-Dedekind sums

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Abstract: In the present paper, the fundamental aim is to consider a $p$-adic continuous function for an odd prime to inside a $p$-adic $q$-analogue of the higher order Extended Dedekind-type sums related to $q$-Genocchi polynomials with weight $\alpha$ by using fermionic $p$-adic invariant $q$-integral on $\mathbb{Z}_p$.

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1 Introduction

Imagine that $p$ be a fixed odd prime number. We now start with definition of the following notations. Let $\mathbb{Q}_p$ be the field $p$-adic rational numbers and let $\mathbb{C}_p$ be the completion of algebraic closure of $\mathbb{Q}_p$.

Thus,

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n : 0 \leq a_n < p \right\}.$$ 

Then $\mathbb{Z}_p$ is integral domain, which is defined by

$$\mathbb{Z}_p = \left\{ x = \sum_{n=0}^{\infty} a_n p^n : 0 \leq a_n < p \right\}$$
or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$ 

We assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ as an indeterminate. The $p$-adic absolute value $|.|_p$ is normally defined by

$$|x|_p = \frac{1}{p^n}$$

where $x = p^n \frac{x}{t}$ with $(p, s) = (p, t) = (s, t) = 1$ and $n \in \mathbb{Q}$ (for details, see [1-19]).

The $p$-adic $q$-Haar distribution is defined by Kim as follows: for any postive integer $n$,

$$\mu_q (a + p^n \mathbb{Z}_p) = (-q)^a \frac{(1 + q)}{1 + q^a}$$

for $0 \leq a < p^n$ and this can be extended to a measure on $\mathbb{Z}_p$ (for details, see [12], [14], [17]).

In [7], the $q$-Genocchi polynomials are defined by Araci et al. as follows:

$$\widetilde{G}_{n,q}^{(\alpha)} (x) = n \int_{\mathbb{Z}_p} \left( \frac{1 - q^{\alpha(x+\xi)}}{1 - q^{\alpha}} \right)^{n-1} d\mu_q (\xi)$$

for $n \in \mathbb{Z}_+ := \{0, 1, 2, 3, \cdots \}$. We easily see that

$$\lim_{q \to 1} \widetilde{G}_{n,q}^{(\alpha)} (x) = G_n (x)$$

where $G_n (x)$ are Genocchi polynomials, which are given in the form:

$$\sum_{n=0}^{\infty} G_n (x) \frac{t^n}{n!} = e^{tx} \frac{2t}{e^t + 1}, \quad |t| < \pi$$

(for details, see [7]). Taking $x = 0$ into (1), then we have $\widetilde{G}_{n,q}^{(\alpha)} (0) := \widetilde{G}_{n,q}^{(\alpha)}$ are called $q$-Genocchi numbers with weight $\alpha$.

The $q$-Genocchi numbers and polynomials have the following identities:

$$\widetilde{G}_{n+1,q}^{(\alpha)} = (n + 1) \frac{1 + q}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^\alpha l + 1},$$

$$\widetilde{G}_{n+1,q}^{(\alpha)} (x) = (n + 1) \frac{1 + q}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1 + q^\alpha l + 1},$$

$$\widetilde{G}_{n,q}^{(\alpha)} (x) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l x} \widetilde{G}_{l,q}^{(\alpha)} \left( \frac{1 - q^{\alpha l}}{1 - q^\alpha} \right)^{n-l}.$$  

Additionally, for $d$ odd natural number, we have

$$\widetilde{G}_{n,q}^{(\alpha)} (dx) = \left( \frac{1 + q}{1 + q^d} \right) \left( \frac{1 - q^{ad}}{1 - q^\alpha} \right)^{n-1} \sum_{a=0}^{d-1} q^a (-1)^a \widetilde{G}_{n,q}^{(\alpha)} (x + \frac{a}{d}),$$

(for details about this subject, see [7]).

For any positive integer $h, k$ and $m$, Dedekind-type DC sums are given by Kim in [9], [10] and [11] as follows:
\[ S_m(h, k) = \sum_{M=1}^{k-1} (-1)^{M-1} \frac{M}{k} \bar{E}_m \left( \frac{hM}{k} \right) \]

where \( \bar{E}_m(x) \) are the \( m \)-th periodic Euler function.

In 2011, Taekyun Kim added a weight to \( q \)-Bernoulli polynomials in [16]. He derived not only new but also interesting properties for weighted \( q \)-Bernoulli polynomials. After, many mathematicians, by utilizing from Kim’s paper [16], have introduced a new concept in Analytic numbers theory as weighted \( q \)-Bernoulli, weighted \( q \)-Euler, weighted \( q \)-Genocchi polynomials in [17], [6], [7], [1], [3] and [5]. Also, the generating function of weighted \( q \)-Genocchi polynomials was introduced by Araci et al. in [7]. They also derived several arithmetic properties for weighted \( q \)-Genocchi polynomials.

Kim has given some interesting properties for Dedekind-type DC sums. He firstly considered a \( p \)-adic continuous function for an odd prime number to contain a \( p \)-adic \( q \)-analogue of the higher order Dedekind-type DC sums in [10].

By the same motivation, we, by using \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \), shall get weighted \( p \)-adic \( q \)-analogue of the higher order Dedekind-type DC sums.

### 2 Extended \( q \)-Dedekind-type sums in connection with \( q \)-Genocchi polynomials with weight \( \alpha \)

If \( x \) is a \( p \)-adic integer, then \( w(x) \) is the unique solution of \( w(x) = w(x)^p \) that is congruent to \( x \) mod \( p \). It can also be defined by

\[ w(x) = \lim_{n \to \infty} x^{p^n}. \]

The multiplicative group of \( p \)-adic units is a product of the finite group of roots of unity, and a group isomorphic to the \( p \)-adic integers. The finite group is cyclic of order \( p-1 \) or \( 2 \), as \( p \) is odd or even, respectively, and so it is isomorphic. Actually, the teichmüller character gives a canonical isomorphism between these two groups.

Let \( w \) be the Teichmüller character (mod \( p \)). For \( x \in \mathbb{Z}_p^* := \mathbb{Z}_p/p\mathbb{Z}_p \), set

\[ \langle x : q \rangle = w^{-1}(x) \left( \frac{1-qx}{1-q} \right). \]

Let \( a \) and \( N \) be positive integers with \( (p, a) = 1 \) and \( p \mid N. \) We now introduce the following

\[ \tilde{E}_q^{(\alpha)}(s, a, N : q^N) = w^{-1}(a) \langle x : q^\alpha \rangle^s \sum_{j=0}^{\infty} \left( \begin{array}{c} s \cr j \end{array} \right) q^{\alpha aj} \left( \frac{1-q^{\alpha N}}{1-q^{\alpha a}} \right)^j \tilde{G}^{(\alpha)}_{j, q^N}. \]

In particular, if \( m+1 \equiv 0(\text{mod} \ p-1) \), then we have

\[ \tilde{E}_q^{(\alpha)}(m, a, N : q^N) = \left( \frac{1-q^{\alpha a}}{1-q^{\alpha}} \right)^m \sum_{j=0}^{m} \left( \begin{array}{c} m \cr j \end{array} \right) q^{\alpha aj} \tilde{G}^{(\alpha)}_{j, q^N} \left( \frac{1-q^{aN}}{1-q^{\alpha a}} \right)^j \]

\[ = \left( \frac{1-q^{aN}}{1-q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left( \frac{1-q^{aN}(\xi+\frac{\alpha}{\alpha})}{1-q^{\alpha a N}} \right)^m d\mu_{q^\alpha}(\xi). \]
Then, $\tilde{E}_q^{(a)}(m, a, N : q^N)$ is a continuous $p$-adic extension of

$$\left(1 - q^{aN}\right) \frac{m}{m+1} \tilde{G}_{m+1,q^N}^{(a)} \left(\frac{a}{N}\right).$$

Suppose that $[.]$ be the Gauss’ symbol and let $\{x\} = x - [x]$. Thus, we are now ready to treat $q$-extension of the higher order Dedekind-type DC sums $\tilde{S}_{m,q}^{(\alpha)}(h, k : q)$ in the form:

$$\tilde{S}_{m,q}^{(\alpha)}(h, k : q) = \sum_{M=1}^{k-1} (-1)^{M-1} \left(1 - q^{\alpha M}\right) \int_{\mathbb{Z}_p} \left(1 - q^{\alpha(i\frac{hM}{k}+\frac{h}{k})}\right)^m \, d\mu_q(\xi).$$

If $m+1 \equiv 0 \pmod{p-1}$,

$$\sum_{M=1}^{k-1} (-1)^{M-1} \left(1 - q^{\alpha M}\right) \left(1 - q^{\alpha k}\right) \int_{\mathbb{Z}_p} \left(1 - q^{\alpha k(hM/k+1)}\right)^m \, d\mu_q(\xi)$$

$$= \sum_{M=1}^{k-1} (-1)^{M-1} \left(1 - q^{\alpha M}\right) \left(1 - q^{\alpha k}\right) \int_{\mathbb{Z}_p} \left(1 - q^{\alpha k(hM/k+1)}\right)^m \, d\mu_q(\xi)$$

where $p \mid k, (hM, p) = 1$ for each $M$. Via the equation (1), we easily state the following

$$\int_{\mathbb{Z}_p} \left(1 - q^{\alpha(x+\xi)}\right)^k \, d\mu_q(\xi)$$

$$= \left(1 - q^{\alpha m}\right)^k \left(1 + q\right) \sum_{i=0}^{m-1} (-1)^i \int_{\mathbb{Z}_p} \left(1 - q^{\alpha m(\xi+i)}\right)^k \, d\mu_q(\xi).$$

Due to (6) and (7), we easily obtain

$$\int_{\mathbb{Z}_p} \left(1 - q^{\alpha N}\right)^m \int_{\mathbb{Z}_p} \left(1 - q^{\alpha N(\xi+\frac{a}{N})}\right)^m \, d\mu_q(\xi)$$

$$= \frac{1 + q^N}{1 + q^{Np}} \sum_{i=0}^{p-1} (-1)^i \left(1 - q^{\alpha Np}\right)^m \int_{\mathbb{Z}_p} \left(1 - q^{\alpha Np(\xi+i)}\right)^m \, d\mu_q(\xi).$$

Thanks to (6), (7) and (8), we discover the following $p$-adic integration:

$$\tilde{E}_q^{(a)}(s, a, N : q^N) = \frac{1 + q^N}{1 + q^{Np}} \sum_{0 \leq i \leq p-1 \atop a + iN \neq 0 \pmod{p}} (-1)^i \tilde{E}_q^{(a)}(s, (a + iN)pN, p^N : q^{pN}).$$

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On the other hand,
\[
\tilde{E}_q^{(\alpha)}(m, a, N : q^N) = \left(1 - q^{\alpha N}\right)^m \int_{\mathbb{Z}_p} \left(1 - q^{\alpha N(\xi + \frac{1}{q})}\right)^m d\mu_{q^N}(\xi)
\]
\[
- \left(1 - q^{\alpha NP}\right)^m \int_{\mathbb{Z}_p} \left(1 - q^{\alpha p^N(\xi + \frac{a + N}{p})}\right)^m d\mu_{q^N}(\xi)
\]
where \((p^{-1}a)_N\) denotes the integer \(x\) with \(0 \leq x < N\), \(px \equiv a\) (mod \(N\)) and \(m\) is integer with \(m + 1 \equiv 0\) (mod \(p - 1\)). Therefore, we can state the following
\[
\sum_{M=1}^{k-1} (-1)^{M-1} \left(1 - q^{\alpha M}\right)^{m+1} \tilde{S}_{m,q}^{(\alpha)}(h, k : q^k) - \left(1 - q^{\alpha k}\right)^{m+1} \left(1 - q^{\alpha kp}\right)^m \tilde{S}_{m,q}^{(\alpha)}((p^{-1}h) : k : q^{pk})
\]
where \(p \nmid k\) and \(p \nmid hm\) for each \(M\). Thus, we obtain the following definition, which seems interesting for further studying in theory of Dedekind sums.

**Definition 2.1** Let \(h, k\) be positive integers with \((h, k) = 1\), \(p \nmid k\). For \(s \in \mathbb{Z}_p\), we define \(p\)-adic Dedekind-type DC sums as follows:
\[
\tilde{S}_{p,q}^{(\alpha)}(s : h, k : q^k) = \sum_{M=1}^{k-1} (-1)^{M-1} \left(1 - q^{\alpha M}\right)^{m+1} \tilde{E}_q^{(\alpha)}(m, hM, k : q^k).
\]

As a result of the above definition, we derive the following theorem.

**Theorem 2.2** For \(m + 1 \equiv 0\) (mod \(p - 1\)) and \((p^{-1}a)_N\) denotes the integer \(x\) with \(0 \leq x < N\), \(px \equiv a\) (mod \(N\)), then, we have
\[
\tilde{S}_{p,q}^{(\alpha)}(s : h, k : q^k) = \left(1 - q^{\alpha k}\right)^{m+1} \tilde{S}_{m,q}^{(\alpha)}(h, k : q^k)
\]
\[
- \left(1 - q^{\alpha k}\right)^{m+1} \left(1 - q^{\alpha kp}\right)^m \tilde{S}_{m,q}^{(\alpha)}((p^{-1}h) : k : q^{pk}).
\]

In the special case \(\alpha = 1\), our applications in theory of Dedekind sums resemble Kim’s results in [10]. These results seem to be interesting for further studies in [9], [11] and [18].

**References**


