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A Note on modified Jacobsthal and Jacobsthal–Lucas numbers

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Abstract: In this note, some relations between Jacobsthal and Jacobsthal–Lucas numbers and their respective modifications due to K. T. Atanassov [1, 2] and Y. Shang [4] are presented.
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1 Introduction

In [3], Rabago introduced the concept of circulant determinant sequence with binomial coefficients. In particular, the right-circulant determinant sequence with binomial coefficients, denoted by $\{R_n\}$, is defined as a sequence of the form

$$\{R_n\} = \left\{ |1|, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \\ 3 & 3 & 1 & 1 \end{vmatrix}, \dots \right\}.$$

Furthermore, in [1], the formula for the n-th term of the sequence R_n, as well as the sum of the first n terms, denoted by RS_n, has been shown by the author and is given by

$$R_n = \left(1 + (-1)^{n-1}\right) 2^{n-2}$$

and

$$RS_n = \frac{4^{\lfloor \frac{n+1}{2} \rfloor} - 1}{3}.$$

As we recall, the n-th Jacobsthal and Jacobsthal-Lucas numbers $(n \geq 0)$ are defined by

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1}$$

and

$$j_n = 2^n + (-1)^n, \quad (2)$$

respectively. Now, if we take the sum of the first n-term of the sequence $\{R_n\}$ with odd indices (or simply the first $(2n-1)$ terms of the sequence) we will obtain

$$RS_{2n-1} = \frac{4^n - 1}{3}.$$

Fortunately, the product of the first n Jacobsthal numbers and Jacobsthal-Lucas numbers is given by the same formula. Hence, we obtain the following result.

Theorem 1.1. *Let J_n, j_n and RS_n be the n-th Jacobsthal number, the n Jacobsthal-Lucas number, and the sum of the first n terms of the circulant determinant sequence with binomial coefficients with odd indices then,*

$$J_n j_n = RS_{2n-1}.$$

On the otherhand, Atanassov [1] provide a generalization of (1) by the following formula

$$J_n^s = \frac{s^n - (-1)^n}{s+1}, \quad (3)$$

where $n \geq 0$ is a natural number and $s \geq 0$ is a real number. He also introduced another generalization of (1) in [2] and is given by

$$Y_n^s = \frac{s^n - (-1)^n}{s^2 - 1}, \quad (4)$$

where $s \neq 1$ is a real number. He then obtained the following interesting results.

Theorem 1.2. *For every natural number $n \geq 0$ and real number $s \neq 1$:*

$$Y_n^s = \frac{1}{s-1} J_n^s,\ Y_{n+2}^s = Y_n^s + s^n,\ Y_{n+1}^s = sY_n^s + \frac{(-1)^n}{s-1}.$$

As an analogue to these results, Shang [4] formulated some modifications of the Jacobsthal-Lucas numbers. More precisely, he consider the following modification of (2):

$$j_n^s = s^n + (-1)^n, \quad (5)$$

where n is a natural number and $s \geq 0$ is a real number. He then further extend his modification to the following form

$$j_n^{s,t} = s^n + (-t)^n, \quad (6)$$

where n is a natural number, s and t are arbitrary real numbers. Inspired by these results, we present some relations involving Jacobsthal numbers, Jacobsthal-Lucas numbers, and their respective generalization and modification.

2 Main results

We begin by defining J^s_{-n} and j^s_{-n}.

For every $n \in \mathbb{N}$, we let

$$J^s_{-n} = (-1)^{n+1} J^s_n \tag{7}$$

and

$$j^s_{-n} = (-1)^{n+1} j^s_n. \tag{8}$$

Throughout the following discussion we let $C^n_k = \begin{pmatrix} n \\ k \end{pmatrix}$.

Theorem 2.1. *For $n, l \in \mathbb{N}$ and real number $s \geq 0$, $s \neq 1$,*

$$\sum_{k=0}^{n} C^n_k j^{s^l}_{2k} = (j^s_{2l})^n + 2^n. \tag{9}$$

In particular, for $l = 1$, we have $\sum_{k=0}^{n} C^n_k j^{s^2}_{2k} = (j^s_2)^n + 2^n$.

Proof. Note that $(s^{2l} + 1)^n = \sum_{k=0}^{n} C^n_k (s^{2l})^k$ and since $\sum_{k=0}^{n} C^n_k = 2^n$ then

$$(j^s_{2l})^n = \sum_{k=0}^{n} C^n_k \left((s^l)^{2k} + 1\right) - 2^n = \sum_{k=0}^{n} C^n_k j^{s^l}_{2k} - 2^n.$$

□

We may remark that in terms of (6), we can express (9) as $j_n^{j^s_{2l}, -2}$.

Theorem 2.2. *For $n, l \in \mathbb{N}$ and real number $s \geq 0$, $s \neq 1$,*

$$\sum_{k=0}^{n} C^n_k j^{s^2}_k = j^{s^2+1}_n - (-1)^n.$$

More generally, we have,

$$\sum_{k=0}^{n} C^n_k j^{s^{2l}}_k = j^{s^{2l}+1}_n - (-1)^n.$$

Proof. Because $(s^{2l} + 1)^n = \sum_{k=0}^{n} C^n_k (s^{2l})^k + \sum_{k=0}^{n} C^n_k (-1)^k$, then

$$(s^{2l} + 1)^n = \sum_{k=0}^{n} C^n_k j^{s^{2l}}_k$$

Hence,

$$j^{s^{2l}+1}_n = \sum_{k=0}^{n} C^n_k j^{s^{2l}}_k + (-1)^n$$

Thus, conclusion follows. □

Theorem 2.3. *For all natural number n and real number $s \geq 0$, $s \neq 1$,*

$$\sum_{k=0}^{n} C^n_k j^{s-1}_k = s^n.$$

Proof. We use the fact that $\sum_{k=0}^{n} C_k^n (-1)^k = 0$. Since $s^n = (s-1+1)^n$ then

$$\begin{aligned}(s-1+1)^n &= \sum_{k=0}^{n} C_k^n (s-1)^k + \sum_{k=0}^{n} C_k^n (-1)^k \\ &= \sum_{k=0}^{n} C_k^n \left((s-1)^k + (-1)^k\right) \\ &= \sum_{k=0}^{n} C_k^n j_k^{s-1}.\end{aligned}$$

Thus, conclusion follows. □

By incorporating Theorem 1.2 to the previous theorem we obtain the following corollaries.

Corollary 2.4. *For all natural number n and real number $s \geq 0$, $s \neq 1$,*

$$Y_{n+2}^s - Y_n^s = \sum_{k=0}^{n} C_k^n j_k^{s-1}.$$

Corollary 2.5. *For all natural number n and real number $s \geq 0$, $s \neq 1$,*

$$J_{n+2}^s - J_n^s = (s-1) \sum_{k=0}^{n} C_k^n j_k^{s-1}.$$

In particular, for $s = 2$, $J_{n+2} - J_n = \sum_{k=0}^{n} C_k^n j_k^1 = 2^n$.

Using the identities in Theorem 2.2 and Theorem 2.3 we will obtain the following theorem.

Theorem 2.6. *For all natural number n and real number $s \geq 0$, $s \neq 1$,*

$$J_n^s = \frac{1}{s+1}\left[\sum_{k=0}^{n} C_k^n \left(j_k^{s^{2l}} + j_k^{s-1}\right) - j_n^{s^{2l}+1}\right] = (s-1)Y_n^s.$$

If we let $s = 2$ we'll obtain the following special case of the above theorem.

Corollary 2.7.

$$J_n = \frac{1}{3}\left[\sum_{k=0}^{n} C_k^n \left(j_k^{4^l} + j_k^1\right) - j_n^{4^l+1}\right].$$

Theorem 2.6 can also be shown using the fact that $\sum_{k=0}^{n} C_k^n j_k^{s^{2l}} = (s^{2l}+1)^n$.

Theorem 2.8. *For all natural number n and real number $s \geq 0$, $s \neq 1$,*

$$J_n = \frac{1}{3}\left[\sum_{k=0}^{n} C_n^k \left(j_k^{s^{2l}} + j_{2k}^{s^l}\right) - \left(j_n^{s^{2l}+1} + (j_{2l}^s)^n\right)\right].$$

Theorem 2.9. *For all natural number n and real number $s \geq 0$, $s \neq 1$,*

$$j_n = \sum_{k=0}^{n} C_n^k \left(j_{2k}^{s^l} - j_k^{s^{2l}}\right) + \left(j_n^{s^{2l}+1} - (j_{2l}^s)^n\right).$$

For the proof of Theorem 2.8 and Theorem 2.9 we may use Theorem 2.1 and 2.2.

Theorem 2.10. *For all natural number n and real number $s \geq 0$, $s \neq 1$,*

$$\sum_{m=0}^{n} \sum_{k=0}^{m} C_k^n j_k^{s-1} = \frac{s^{n+1} - 1}{s - 1}.$$

Proof. Use Theorem 2.3. □

Theorem 2.11. *For all even natural number n,*

$$J_n = \frac{1}{3} \left[\left(\sum_{k=0}^{n} (-1)^k C_k^n j_k^s \right) - j_n^{s-1} \right],$$

where J_n is the n-th Jacobsthal number and j_n^s is the n-th modified Jacobsthal-Lucas number.

Proof. The proof is straightforward. We use the binomial expansion

$$(x + y)^n = \sum_{k=0}^{n} C_k^n x^k y^{n-k}.$$

We let $x = s$ and $y = -1$ obtaining

$$(s - 1)^n = \sum_{k=0}^{n} (-1)^{n-k} C_k^n s^n.$$

Noting that $(-1)^{n-k} = (-1)^{n+k}$, we have

$$(s - 1)^n = \sum_{k=0}^{n} (-1)^{n+k} C_k^n s^k + \sum_{k=0}^{n} C_k^n - 2^n.$$

Adding $(-1)^n$ both sides we'll obtain

$$j_n^{s-1} = (s - 1)^n + (-1)^n = \sum_{k=0}^{n} C_k^n ((-1)^{n+k} s^k + 1) - (2^n - (-1)^n),$$

and since n is even then,

$$j_n^{s-1} = \sum_{k=0}^{n} (-1)^k C_k^n (s^k + (-1)^k) - 3J_n.$$

Thus,

$$J_n = \frac{1}{3} \left[\left(\sum_{k=0}^{n} (-1)^k C_k^n j_k^s \right) - j_n^{s-1} \right],$$

which is the desired result. □

A similar result to Theorem 2.11 can be obtain for j_n and is stated in the following theorem.

Theorem 2.12. *For all odd natural number n,*

$$j_n = j_n^{s-1} + \sum_{k=0}^{n}(-1)^k C_k^n j_k^s, \tag{10}$$

where j_n is the n-th Jacobsthal-Lucas number and j_n^s is the n-th modified Jacobsthal-Lucas number.

Proof. The proof is similar to the previous theorem. Again we let $x = s$ and $y = -1$ in the binomial expansion obtaining

$$(s-1)^n = \sum_{k=0}^{n}(-1)^{n-k} C_k^n s^n.$$

So we have,

$$(s-1)^n = \sum_{k=0}^{n}(-1)^{n+k} C_k^n s^k - \sum_{k=0}^{n} C_k^n + 2^n.$$

Adding both sides by $(-1)^n$ and noting that n is odd we have

$$j_n^{s-1} = -\sum_{k=0}^{n}(-1)^k C_k^n (s^k + (-1)^k) + j_n,$$

It follows that,

$$j_n = j_n^{s-1} + \sum_{k=0}^{n}(-1)^k C_k^n j_n^s.$$

This proves the theorem. □

We could express (10) using (8) and is given by the following Corollary.

Corollary 2.13. $j_n = j_n^{s-1} - \sum_{k=0}^{n} C_k^n j_{-k}^s$.

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