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On the mean values of Dedekind sums over short intervals

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Abstract: The main purpose of this paper is using the mean values of Dirichlet *L*-functions and estimates for character sums to study the mean values of Dedekind sums over short intervals.

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1 Introduction and Main results

For a positive integer k and arbitrary integer h, the classical Dedekind sum $S(h, k)$ is defined by

$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right)$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer,}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

and $[x]$ is the integral part of x.

This sum was first studied by Dedekind because of the prominent role it plays in the transformation theory of the Dedekind η-function, which is a modular form of weight 1/2 for the full modular group $SL_2(Z)$. One of the most famous properties of the Dedekind sum is the reciprocity formula

$$S(h,k)+S(k,h)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4},\quad (h,k)=1.$$

H. Walum [1] has shown that for prime k,

$$\sum_{h=1}^{k}|S(h,k)|^2=\frac{1}{\pi^4}\frac{k^2}{\varphi(k)}\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}}|L(1,\chi)|^4,$$

where $\varphi(k)$ is the Euler function and $L(s,\chi)$ denotes the Dirichlet *L*-function corresponding to $\chi \bmod k$. J. B. Conrey, E. Fransen, R. Klein and C. Scott [2] studied the mean value of Dedekind sums and proved the following:

Proposition 1.1. *Suppose that m is a given positive integer and k is any sufficiently large integer. Then,*

$$\sum_{h=1}^{k}{}' S^{2m}(h,k) = f_m(k)\left(\frac{k}{12}\right)^{2m} + O((k^{9/5} + k^{2m-1+1/(m+1)})\log^3 k),$$

where $f_m(k)$ is defined by the Dirichlet series

$$\sum_{k=1}^{\infty}\frac{f_m(k)}{k^s} = \frac{\zeta^2(2m)}{\zeta(4m)}\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s),$$

$\zeta(s)$ is the Riemann zeta function and Σ' denotes the summation over all integers that are coprime with k.

C. Jia [3] improved the error terms in Proposition 1.1. In the spirit of [2] and [4], W. Zhang [5] used an estimate for character sums to prove the following:

Proposition 1.2. *Suppose that p is any sufficiently large prime number and n is any positive integer. Then, for $k = p^n$, we have*

$$\sum_{h=1}^{k}{}' |S(h,k)|^2 = \frac{5k^2}{144}\frac{(p^2-1)^2}{p(p^3-1)} + O\left(k\exp\left(\frac{3\log k}{\log\log k}\right)\right)$$

where the O-constant is absolute.

Let c and d be an integers, $c>0$. The Hardy sums, which are usually regarded analogues to the classical Dedekind sums, are defined by

$$H(d,c) = \sum_{j=1}^{c-1}(-1)^{j+1+[dj/c]}.$$

Z. Xu and W. Zhang [6] studied the mean value

$$\sum_{a\le p/4}\sum_{b\le p/4} H(2a\overline{b},p)$$

for any prime $p\ge 5$, and gave some sharp asymptotic formulae.

How about the mean value of $S(h,k)$ when h runs through an interval $[1,\lambda k]$, $\lambda\in(0,1)$? It seems difficult to obtain an asymptotic formula even in cases such as $\lambda=1/2$ or $1/4$. Here we study the better behaved mean value:

$$\sum_{a\le k/2}{}'\sum_{b\le k/2}{}' S(a\overline{b},k), \tag{1.1}$$

$$\sum_{a\le k/4}{}'\sum_{b\le k/4}{}' S(a\overline{b},k), \tag{1.2}$$

where $b\overline{b}\equiv 1 (\bmod k)$.

The main purpose of this paper is using the mean values of Dirichlet L-functions and estimates for character sums to study the mean values of Dedekind sums of the types (1.1) and (1.2). Namely, we have the followings

Theorem 1.3. *Let* p *be an odd prime,* $k = p^{\alpha}$, $\alpha \ge 1$. *Then we have*

$$\sum_{a\le k/2}{}' \sum_{b\le k/2}{}' S(a\bar{b},k) = \frac{1}{16}k^2 \frac{(p^2-1)^2}{(p^2+1)(p^2+p+1)} + O(k^{1+\varepsilon}).$$

Theorem 1.4. *Let* p *be an odd prime,* $k = p^{\alpha}$, $\alpha \ge 1$. *Then we have*

$$\sum_{a\le k/4}{}' \sum_{b\le k/4}{}' S(a\bar{b},k) = \frac{3k^2}{64} \frac{(p^3+1)(p^2-1)(p-1)}{(p^2+1)(p^4+p^2+1)} + O(k^{1+\varepsilon}).$$

Taking $k = p$ as a prime in Theorem 1.3 and Theorem 1.4, respectively, then we have the following:

Corollary 1.5. *Let* p *be an odd prime. Then we have*

$$\sum_{a\le p/2} \sum_{b\le p/2} S(a\bar{b},p) = \frac{1}{16}p^2 + O(p^{1+\varepsilon}).$$

Corollary 1.6. *Let* $p \ge 5$ *be a prime. Then we have*

$$\sum_{a\le p/4} \sum_{b\le p/4} S(a\bar{b},p) = \frac{3}{64}p^2 + O(p^{1+\varepsilon}).$$

In fact, in his doctorial thesis, Z. Xu obtained the result in Corollary 1.6 by analytic method, and it not very easy to obtain Theorem 1.3 and Theorem 1.4 by his approach. Our proof here seems to be effective and more elementary.

2 Lemmas

To complete the proofs of the theorems, we need the following lemmas.

Lemma 2.1. Let $k \ge 3$ be an integer and $(h,k)=1$. Then

$$S(h,k) = \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(h)\,|L(1,\chi)|^2.$$

Proof. See[5].

Lemma 2.2. Let k be an odd integer, and χ denote a Dirichlet character modulo k with $\chi(-1)=-1$. Then,

$$\sum_{a\le k/2}\chi(a) = \frac{\bar{\chi}(2)-2}{k}\sum_{a\le k} a\chi(a), \quad (k\ge 3) \tag{2.1}$$

$$\sum_{a\le k/4}\chi(a) = \frac{\bar{\chi}(4)-\bar{\chi}(2)-2}{2k}\sum_{a\le k} a\chi(a), \quad (k\ge 5) \tag{2.2}$$

$$\sum_{a\le k/4}\chi(a+k/4)=\frac{-\overline{\chi}(4)+3\overline{\chi}(2)-2}{2k}\sum_{a\le k}a\chi(a),\quad (k\ge 5) \tag{2.3}$$

Proof. The proofs of (2.1) and (2.2) can be found in [6] and [7]. And (2.3) follows immediately from (2.1) and (2.2).

Lemma 2.3. Let p be an odd prime, $k=p^{\alpha}$, $\alpha\ge 1$. Then, for any $d\mid k$, $d>1$, $\chi \bmod d$ with $\chi(-1)=-1$, we have

$$\left|\sum_{a\le k/2}\chi(a)\right|^2=\frac{1}{d^2}(5-2\overline{\chi}(2)-2\chi(2))\left|\sum_{a\le d}a\chi(a)\right|^2.$$

Proof. For any $d\mid k, d>1, \chi \bmod d$ with $\chi(-1)=-1$, we have

$$\begin{aligned}\sum_{a\le k/2}\chi(a)&=\sum_{0\le n\le k/d-1}\ \sum_{nd/2\le a\le (n+1)d/2}\chi(a)\\&=\sum_{0\le n\le k/d-1}\ \sum_{a\le d/2}\chi(a+nd/2)\\&=\sum_{\substack{0\le n\le k/d-1\\ n\ \text{even}}}\ \sum_{a\le d/2}\chi(a)+\sum_{\substack{0\le n\le k/d-1\\ n\ \text{odd}}}\ \sum_{a\le d/2}\chi(a+d/2)\\&=\sum_{\substack{0\le n\le k/d-1\\ n\ \text{even}}}\ \sum_{a\le d/2}\chi(a)+\sum_{\substack{0\le n\le k/d-1\\ n\ \text{odd}}}\ \sum_{a\le d/2}\chi(d-a)\\&=\sum_{\substack{0\le n\le k/d-1\\ n\ \text{even}}}\ \sum_{a\le d/2}\chi(a)-\sum_{\substack{0\le n\le k/d-1\\ n\ \text{odd}}}\ \sum_{a\le d/2}\chi(a)\end{aligned}$$

Since p is an odd prime, then k/d is odd, thus

$$\sum_{a\le k/2}\chi(a)=\sum_{a\le d/2}\chi(a).$$

By Lemma 2.2, we have

$$\begin{aligned}\left|\sum_{a\le k/2}\chi(a)\right|^2&=\left|\sum_{a\le d/2}\chi(a)\right|^2\\&=\left|\frac{\overline{\chi}(2)-2}{d}\sum_{a\le d}a\chi(a)\right|^2\\&=\frac{1}{d^2}\left(5-2\overline{\chi}(2)-2\chi(2)\right)\left|\sum_{a\le d}a\chi(a)\right|^2.\end{aligned}$$

Then, the lemma is proved. □

Lemma 2.4. Let p be an odd prime, $k=p^{\alpha},\alpha\ge 1$. Then for any $d\mid k$, $d>1$, $\chi \bmod d$ with $\chi(-1)=-1$, we have

$$\left|\sum_{a\le k/4}\chi(a)\right|^2=\frac{\delta}{4d^2}\left|\sum_{a\le d}a\chi(a)\right|^2,$$

where

$$\delta=\begin{cases}6-4\overline{\chi}(4)-2\chi(4)+\overline{\chi}(2)+\chi(2), & \text{if } p\equiv 1(\bmod 4) \text{ or } p\equiv 3(\bmod 4), k/d\equiv 1(\bmod 4),\\ 14+2\overline{\chi}(4)+2\chi(4)-9\overline{\chi}(2)-9\chi(2), & \text{if } p\equiv 3(\bmod 4), k/d\equiv 3(\bmod 4).\end{cases}$$

Proof. For any $d\mid k, d>1, \chi \bmod d$ with $\chi(-1)=-1$, we have

$$\begin{aligned}\sum_{a\le k/4}\chi(a) &= \sum_{0\le n\le k/d-1}\ \sum_{nd/4<a\le(n+1)d/4}\chi(a)\\ &= \sum_{0\le n\le k/d-1}\ \sum_{a\le d/4}\chi(a+nd/4)\\ &= \sum_{\substack{0\le n\le k/d-1\\ n\equiv 0(\bmod 4)}}\ \sum_{a\le d/4}\chi(a)+\sum_{\substack{0\le n\le k/d-1\\ n\equiv 1(\bmod 4)}}\ \sum_{a\le d/4}\chi(a+d/4)\\ &\quad+\sum_{\substack{0\le n\le k/d-1\\ n\equiv 2(\bmod 4)}}\ \sum_{a\le d/4}\chi(a+d/2)+\sum_{\substack{0\le n\le k/d-1\\ n\equiv 3(\bmod 4)}}\ \sum_{a\le d/4}\chi(a+3d/4)\end{aligned}$$

Since

$$\begin{aligned}\sum_{a\le d/4}\chi(a+d/2) &= \sum_{a\le d/4}\chi(3d/4-a)=\sum_{a\le d/4}\chi(-d/4-a)=-\sum_{a\le d/4}\chi(a+d/4)\\ &= \sum_{a\le d/4}\chi(d-a)=\sum_{a\le d/4}\chi(-a)=-\sum_{a\le d/4}\chi(a)\end{aligned}$$

thus

$$\sum_{a\le k/4}\chi(a)=\varepsilon_1\sum_{a\le d/4}\chi(a)+\varepsilon_2\sum_{a\le d/4}\chi(a+d/4),$$

where

$$\varepsilon_1=\sum_{\substack{0\le n\le k/d-1\\ n\equiv 0(\bmod 4)}}1-\sum_{\substack{0\le n\le k/d-1\\ n\equiv 3(\bmod 4)}}1,$$

$$\varepsilon_2=\sum_{\substack{0\le n\le k/d-1\\ n\equiv 1(\bmod 4)}}1-\sum_{\substack{0\le n\le k/d-1\\ n\equiv 2(\bmod 4)}}1.$$

Now we need to determine the values of ε_1 and ε_2:

If $p\equiv 1(\bmod 4)$, then for any $s>0$, we have $p^s\equiv 1(\bmod 4)$. If $p\equiv 3(\bmod 4)$, then for any $s>0$, we have

$$p^s\equiv\begin{cases}1\ (\bmod 4), & \text{if } s\equiv 0\ (\bmod 2),\\ 3\ (\bmod 4), & \text{if } s\equiv 1\ (\bmod 2).\end{cases}$$

Thus if $p\equiv 1(\bmod 4)$, then $k/d\equiv 1(\bmod 4)$, so $\varepsilon_1=1$, $\varepsilon_2=0$. If $p\equiv 3(\bmod 4)$, then

$$(\varepsilon_1,\varepsilon_2)=\begin{cases}(1,0), & \text{if } k/d\equiv 1\ (\bmod 4),\\ (0,1), & \text{if } k/d\equiv 3\ (\bmod 4).\end{cases}$$

Hence, the problem can be divided into two cases：

(i) $p\equiv 1(\bmod 4)$ or $p\equiv 3(\bmod 4), k/d\equiv 1(\bmod 4)$,

(ii) $p\equiv 3(\bmod 4)$, $k/d\equiv 3(\bmod 4)$.

So

$$\sum_{a\le k/4}\chi(a)=\begin{cases}\sum_{a\le d/4}\chi(a), & \text{if the case (i) holds，}\\ \sum_{a\le d/4}\chi(a+d/4), & \text{if the case (ii) holds.}\end{cases}\tag{2.4}$$

Combining Lemma 2.2 and (2.4), we have

$$\sum_{a\le k/4}\chi(a)=\begin{cases}\dfrac{\overline{\chi}(4)-\overline{\chi}(2)-2}{2d}\sum_{a\le d}a\chi(a), & \text{if the case (i) holds,}\\ \dfrac{-\overline{\chi}(4)+3\overline{\chi}(2)-2}{2d}\sum_{a\le d}a\chi(a), & \text{if the case (ii) holds,}\end{cases}$$

from which one can obtain that in the case (i),

$$\begin{aligned}\left|\sum_{a\le k/4}\chi(a)\right|^2&=\left|\frac{\overline{\chi}(4)-\overline{\chi}(2)-2}{2d}\sum_{a\le d}a\chi(a)\right|^2\\&=\frac{1}{4d^2}\left(6-2\overline{\chi}(4)-2\chi(4)+\overline{\chi}(2)+\chi(2)\right)\left|\sum_{a\le d}a\chi(a)\right|^2,\end{aligned}$$

and in the case (ii),

$$\begin{aligned}\left|\sum_{a\le k/4}\chi(a)\right|^2&=\left|\frac{-\overline{\chi}(4)+3\overline{\chi}(2)-2}{2d}\sum_{a\le d}a\chi(a)\right|^2\\&=\frac{1}{4d^2}\left(14+2\overline{\chi}(4)+2\chi(4)-9\overline{\chi}(2)-9\chi(2)\right)\left|\sum_{a\le d}a\chi(a)\right|^2.\end{aligned}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. Let χ be a primitive character modulo q with $\chi(-1)=-1$. Then we have

$$\frac{1}{q}\sum_{a\le q}a\chi(a)=\frac{i}{\pi}\tau(\chi)L(1,\overline{\chi}),$$

where $\tau(\chi)=\sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right)$ is the Gauss sum, $e(y)=e^{2\pi iy}$.

Proof. This can be easily deduced from Theorems 12.11 and 12.20 of [8]. □

Lemma2.6. Let p be an odd prime and $\alpha>0, q=p^{\alpha}, m\ge 0$. The we have

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(2^m)\left|L(1,\chi)\right|^2\left|\sum_{a\le q}a\chi(a)\right|^2=\frac{(3m+5)\pi^2}{72\cdot 2^{m+1}}q^4\,\frac{(p-1)^4(p+1)^2}{p^4(p^2+1)}+O(q^{3+\varepsilon}).$$

Proof. For each $\chi \bmod q$ (q arbitrary), it is clear that there exists one and only one $d\mid q$ with a unique primitive character $\chi_1 \bmod d$, such that $\chi=\chi_1\chi^0$, where χ^0 denotes the principal character mod q. We have

$$\begin{aligned}&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(2^m)\left|L(1,\chi)\right|^2\left|\sum_{a\le q}a\chi(a)\right|^2\\&=\sum_{d\mid q}\ \sideset{}{^*}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi\chi^0(2^m)\left|L(1,\chi\chi^0)\right|^2\left|\sum_{a\le q}a\chi\chi^0(a)\right|^2,\end{aligned}$$

where Σ^* denotes the summation over all primitive characters mod *d*. Since $q=p^{\alpha}$, $d\mid q$, then for any integer a, $(a,q)=1$ is equivalent to $(a,d)=1$. Thus,

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\chi(2^m)\left|L(1,\chi)\right|^2\left|\sum_{a\le q}a\chi(a)\right|^2\\
&=\sum_{d\mid q}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\chi(2^m)\left|L(1,\chi)\right|^2\left|\sum_{a\le q}a\chi(a)\right|^2\\
&=q^2\sum_{d\mid q}\frac{1}{d^2}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\chi(2^m)\left|L(1,\chi)\right|^2\left|\sum_{a\le d}a\chi(a)\right|^2.
\end{aligned}\tag{2.5}$$

From (2.5), Lemma 2.5 and the identity $|\tau(\chi)|=\sqrt{d}$, for a primitive character $\chi \bmod d$, we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\chi(2^m)\left|L(1,\chi)\right|^2\left|\sum_{a\le q}a\chi(a)\right|^2\\
&=\frac{q^2}{\pi^2}\sum_{d\mid q}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\chi(2^m)\left|L(1,\chi)\right|^2\left|\tau(\chi)L(1,\overline{\chi})\right|^2\\
&=\frac{q^2}{\pi^2}\sum_{d\mid q}d\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\chi(2^m)\left|L(1,\chi)\right|^4.
\end{aligned}\tag{2.6}$$

In [4], it is proved that for any integer $m\ge 0$,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\chi(2^m)\left|L(1,\chi)\right|^4=\frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}J(q)\prod_{p\mid q}\frac{(p^2-1)^3}{p^4(p^2+1)}+O(q^{\varepsilon}),\tag{2.7}$$

where $J(q)=\sum_{d\mid q}\mu(d)\varphi\left(\frac{q}{d}\right)$ denotes the number of primitive characters modulo q.

From (2.6) and (2.7) one can obtain that

$$\begin{aligned}
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(2^m)\left|L(1,\chi)\right|^2\left|\sum_{a\le q}a\chi(a)\right|^2&=\frac{q^2}{\pi^2}\sum_{d\mid q}d\left(\frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}J(q)\prod_{p\mid d}\frac{(p^2-1)^3}{p^4(p^2+1)}+O(d^{\varepsilon})\right)\\
&=\frac{(3m+5)\pi^2}{72\cdot 2^{m+1}}q^2\sum_{d\mid q}dJ(q)\prod_{p\mid d}\frac{(p^2-1)^3}{p^4(p^2+1)}+O(q^{3+\varepsilon})\\
&=\frac{(3m+5)\pi^2}{72\cdot 2^{m+1}}q^4\frac{(p-1)^4(p+1)^2}{p^4(p^2+1)}+O(q^{3+\varepsilon}).
\end{aligned}$$

Then the lemma is proved. □

3 Proof of theorems

In this section, we complete the proofs of Theorem 1.3 and Theorem 1.4. Applying Lemma 2.1, Lemma 2.3, Lemma 2.5 and Lemma 2.6, we have

$$\sum_{a\le k/2}{}' \sum_{b\le k/2}{}' S(a\bar{b},k) = \frac{1}{\pi^2 k}\sum_{a\le k/2}{}'\sum_{b\le k/2}{}'\sum_{d|k}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\chi(a\bar{b})\left|L(1,\chi)\right|^2$$

$$= \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\left|L(1,\chi)\right|^2\left|\sum_{a\le k/2}\chi(a)\right|^2$$

$$= \frac{1}{\pi^2 k}\sum_{d|k}\frac{1}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\left|L(1,\chi)\right|^2(5-4\chi(2))\left|\sum_{a\le d}a\chi(a)\right|^2$$

$$= \frac{1}{16}k^2\frac{(p^2-1)^2}{(p^2+1)(p^2+p+1)} + O(k^{1+\varepsilon}).$$

Now we turn to prove Theorem 1.4. Applying Lemma 2.1, we can obtain that

$$\sum_{a\le k/4}{}' \sum_{b\le k/4}{}' S(a\bar{b},k) = \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\left|L(1,\chi)\right|^2\left|\sum_{a\le k/4}\chi(a)\right|^2.$$

If $p\equiv 1 (\bmod 4)$, then

$$\sum_{a\le k/4}{}' \sum_{b\le k/4}{}' S(a\bar{b},k)$$

$$= \frac{1}{4\pi^2 k}\sum_{d|k}\frac{1}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\left|L(1,\chi)\right|^2(6-2\chi(\bar{4})-2\chi(4)+\bar{\chi}(2)+\chi(2))\left|\sum_{a\le d}a\chi(a)\right|^2$$

$$= \frac{3}{64}k^2\frac{(p^2-1)^2}{(p^2+1)(p^2+p+1)} + O(k^{1+\varepsilon}).$$

Suppose $k/d = p^s$ $(s\ge 0)$. If $p\equiv 3(\bmod 4)$, then $k/d\equiv 1(\bmod 4)$ is equivalent to $s\equiv 0(\bmod 2)$, $k/d\equiv 3(\bmod 4)$ and is equivalent to $s\equiv 1(\bmod 2)$, Thus,

$$\sum_{a\le k/4}{}' \sum_{b\le k/4}{}' S(a\bar{b},k) = \frac{1}{\pi^2 k}\sum_{\substack{d|k\\ k/d\equiv 1(\bmod 4)}}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\left|L(1,\chi)\right|^2\left|\sum_{a\le k/4}\chi(a)\right|^2$$

$$+ \frac{1}{\pi^2 k}\sum_{\substack{d|k\\ k/d\equiv 3(\bmod 4)}}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}\left|L(1,\chi)\right|^2\left|\sum_{a\le k/4}\chi(a)\right|^2$$

$$= \sum{}^{1} + \sum{}^{3},$$

where Σ^i denotes the summation over all divisors d of k with $k/d\equiv i(\bmod 4)$.

Since

$$\sum_{d|k}{}^{1}\frac{d^4}{\varphi(d)} = \sum_{\substack{d|k\\ k/d\equiv 1(\bmod 4)}}\frac{d^4}{\varphi(d)} = \sum_{\substack{s=0\\ s \text{ even}}}^{\alpha-1}\frac{p^{4\alpha-4s}}{\varphi(p^{\alpha-s})} = \sum_{s=0}^{\left[\frac{\alpha-1}{2}\right]}\frac{p^{4\alpha-8s}}{\varphi(p^{\alpha-2s})} = \frac{k^3p^7}{(p-1)(p^6-1)} + O(1),$$

$$\sum_{d|k}{}^{3}\frac{d^4}{\varphi(d)} = \sum_{\substack{d|k\\ k/d\equiv 3(\bmod 4)}}\frac{d^4}{\varphi(d)} = \sum_{\substack{s=0\\ s \text{ odd}}}^{\alpha-1}\frac{p^{4\alpha-4s}}{\varphi(p^{\alpha-s})} = \sum_{s=0}^{\left[\frac{\alpha}{2}\right]}\frac{p^{4\alpha+4-8s}}{\varphi(p^{\alpha+1-2s})} = \frac{k^3p^4}{(p-1)(p^6-1)} + O(1),$$

thus

$$\sum{}^{1} = \frac{1}{4\pi^2 k}\sum_{d|k}{}^{1}\frac{1}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}|L(1,\chi)|^2(6-2\chi(\bar{4})-2\chi(4)+\bar{\chi}(2)+\chi(2))\left|\sum_{a\le d}a\chi(a)\right|^2$$

$$= \frac{3}{64}\frac{(p-1)^4(p+1)^2}{p^4(p^2+1)}\sum_{d|k}{}^{1}\frac{d^4}{\varphi(d)}+O(k^{1+\varepsilon})$$

$$= \frac{3k^2}{64}\frac{p^3(p^2-1)(p-1)}{(p^2+1)(p^4+p^2+1)}+O(k^{1+\varepsilon}),$$

$$\sum{}^{3} = \frac{1}{4\pi^2 k}\sum_{d|k}{}^{3}\frac{1}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}{}^{*}|L(1,\chi)|^2(14+2\bar{\chi}(4)+2\chi(4)-9\bar{\chi}(2)-9\chi(2))\left|\sum_{a\le d}a\chi(a)\right|^2$$

$$= \frac{3}{64}\frac{(p-1)^4(p+1)^2}{p^4(p^2+1)}\sum_{d|k}{}^{3}\frac{d^4}{\varphi(d)}+O(k^{1+\varepsilon})$$

$$= \frac{3k^2}{64}\frac{(p^2-1)(p-1)}{(p^2+1)(p^4+p^2+1)}+O(k^{1+\varepsilon}).$$

So we have

$$\sum_{a\le k/4}{}'\sum_{b\le k/4}{}'S(a\bar{b},k) = \frac{3k^2}{64}\frac{(p^3+1)(p^2-1)(p-1)}{(p^2+1)(p^4+p^2+1)}+O(k^{1+\varepsilon}).$$

This completes the proofs of the theorems. □

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