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Fibonacci and Lucas primes

J. V. Leyendekkers[1] and A. G. Shannon[2]

[1] Faculty of Science, The University of Sydney
NSW 2006, Australia

[2] Faculty of Engineering & IT, University of Technology Sydney
NSW 2007, Australia

e-mails: tshannon38@gmail.com, Anthony.Shannon@uts.edu.au

Abstract. The structures of Fibonacci numbers, F_n, formed when n equals a prime, p, are analysed using the modular ring Z_5, Pascal's Triangle as well as various properties of the Fibonacci numbers to calculate "Pascal-Fibonacci" numbers to test primality by demonstrating the many structural differences between the cases when F_n is prime or composite.

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1 Introduction

The Fibonacci numbers have been studied since the thirteenth century [5], though the Fibonacci sequence was actually recorded before 200 BC by Pingala, an Indian Sanskrit grammarian, in his book, Chandahsastra [5], and some of their implicit properties seem to have been known in classical Greek times [3]. Mathematically, they are specified by the initial conditions $F_1 = F_2 = 1$ and the second order homogeneous linear recurrence relation

$$F_{n+1} = F_n + F_{n-1}. \tag{1.1}$$

When the initial conditions are changed to 1 and 3, we get the sequence of Lucas numbers $\{L_n\}$ [8]. This equation is characterised by the ordered Fibonacci and Lucas triples $\{F_n, F_{n+1}, F_{n-1}\}$ and $\{L_n, L_{n+1}, L_{n-1}\}$ which we shall use in the analysis in this paper. We previously [6] found very regular patterns in the structure of the Fibonacci sequence within the modular ring Z_5 (Table 1).

Class	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{4}_5$
Row	$5r_0$	$5r_1+1$	$5r_2+2$	$5r_3+3$	$5r_4+4$
0	0	1	2	3	4
1	5	6	7	8	9
2	10	11	12	13	14
3	15	16	17	18	19
4	20	21	22	23	24
5	25	26	27	28	29

Table 1. The modular ring Z_5

Note in Table 1 that the elements of the set of right-end digits

$$N^* = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9)\}$$

indicate the class. REDs are indicated by an asterisk. The focus in this paper will be on the positional indicator, n, when n is prime, to provide useful information on F_n itself (Table 2).

n	Characteristics of F_n
$3k$	$2 \mid F_n$
$4k$	$3 \mid F_n$
$5k$	$5 \mid F_n$
$6k$	$8 \mid F_n$
prime	F_n can be prime
composite	F_n cannot be prime*
$p^* = 3, 7$	$p \mid F_{p+1}$
$p^* = 1, 9$	$p \mid F_{p-1}$

Table 2. Some characteristics of F_n
(* except $n = 4$)

2 Primitive Fibonacci triples

The triples (F_{p+1}, F_p, F_{p-1}) can be reduced to a primitive form as follows:

- When $p^* \in \{1, 9\}$, then $p \mid F_{p-1}$;
- When $p^* \in \{3, 7\}$, then $p \mid F_{p+1}$.

With $p \mid F_{p-1}$, both F_p and F_{p+1} may be reduced to ($pK \pm 1$), $K \in$ Z. Then, if we use (1.1) factors may be eliminated and the equation simplified (Table 4). The same applies for $p \mid F_{p+1}$ where F_p and F_{p-1} have the form ($pK \pm 1$). For example, for $p = 43$,

$$F_p + 1 = 43(2 \times 3 \times 13 \times 307 \times 421),$$

$$F_{p+1} = 43(3 \times 89 \times 199 \times 307),$$

$$F_{p-1} - 1 = 43(3 \times 3 \times 5 \times 11 \times 41 \times 307).$$

If we eliminate the common factors 3, 43 and 307, then we get the primitive triple:

$$F_p^{/} = 10946 = F_{21} = F_{\frac{p-1}{2}}, \quad F_{p+1}^{/} = 17711 = F_{22} = F_{\frac{p+1}{2}}, \quad F_{p-1}^{/} = 6765 = F_{20} = F_{\frac{p-3}{2}}.$$

When a primitive represents a Lucas triple, it corresponds to a Fibonacci sum; e.g.,

$$F_p^{/} = F_n + F_{n-2}, \quad F_{p+1}^{/} = F_{n+1} + F_{n-1}, \quad F_{p-1}^{/} = F_{n-1} + F_{n-3}.$$

- for $p^* = 3, 7$, $n = \frac{p+1}{2}$, and
- for $p^* = 1, 9$, $n = \frac{p+3}{2}$.

When the primitives are given by $\left(F_n, F_{n+1}, F_{n-1}\right)$, $\left(L_n, L_{n+1}, L_{n-1}\right)$:

- for p^* = 3, 7, $n = \dfrac{p-1}{2}$, and
- for p^* = 1, 9, $n = \dfrac{p+1}{2}$.

Moreover, the last two digits of p indicate whether the primitive represents Fibonacci or Lucas triples. For instance, when p^* = 3, 7, (odd, odd) gives Lucas and (even, odd) gives Fibonacci, whereas when p^* = 1, 9, (even, odd) gives Lucas and (odd, odd) gives Fibonacci. Note also that when p_1 and p_2 are twin primes, the primitives $F^{/}_{p_1}, F^{/}_{p_2}$ equal alternative Lucas / Fibonacci equivalents as in Table 3.

Twin primes	Category	Sequences
11, 13	Prime / prime	Fibonacci / Lucas
17, 19	Prime / composite	Lucas / Fibonacci
29, 31	Prime / composite	Lucas / Fibonacci
41, 43	Composite / prime	Lucas / Fibonacci
59, 61	Prime / prime	Fibonacci / Lucas

Table 3. Twin prime effects

Primality of F_p	***p***	**F_p, F_{p+1}, F_{p-1}**	**Triple Class Structure**	**Primitive Fibonacci & Lucas Triples**	**& Class Structure in Z_5**
yes	11	89 144 55	$\bar{4}_5\ \bar{4}_5\ \bar{0}_5$	8, 13, 5 $\left(F_6, F_7, F_5\right)$	$\bar{3}_5\ \bar{3}_5\ \bar{0}_5$
yes	13	233 377 144	$\bar{3}_5\ \bar{2}_5\ \bar{4}_5$	18, 29, 11 $\left(L_6, L_7, L_5\right)$	$\bar{4}_5\ \bar{4}_5\ \bar{1}_5$
yes	17	1597 2584 987	$\bar{2}_5\ \bar{4}_5\ \bar{2}_5$	47, 76, 29 $\left(L_8, L_9, L_7\right)$	$\bar{2}_5\ \bar{1}_5\ \bar{4}_5$
no	19	4181 6765 2584	$\bar{1}_5\ \bar{0}_5\ \bar{4}_5$	55, 89, 34 $\left(F_{10}, F_{11}, F_9\right)$	$\bar{0}_5\ \bar{4}_5\ \bar{4}_5$
yes	23	28657 46368 17711	$\bar{2}_5\ \bar{3}_5\ \bar{1}_5$	89, 144, 55 $\left(F_{11}, F_{12}, F_{10}\right)$	$\bar{4}_5\ \bar{4}_5\ \bar{0}_5$
yes	29	514229 832040 317811	$\bar{4}_5\ \bar{0}_5\ \bar{1}_5$	1364, 2207, 843 $\left(L_{15}, L_{16}, L_{14}\right)$	$\bar{4}_5\ \bar{2}_5\ \bar{3}_5$
no	31	1346269 2178309 832040	$\bar{4}_5\ \bar{4}_5\ \bar{0}_5$	987, 1597, 610 $\left(F_{16}, F_{17}, F_{15}\right)$	$\bar{2}_5\ \bar{2}_5\ \bar{0}_5$

(table continues)

Primality of F_p	p	F_p, F_{p+1}, F_{p-1}	Triple Class Structure	Primitive Fibonacci & Lucas Triples	& Class Structure in Z_5
no	37	24157817 39088169 14930352	$\bar{2}_5\ \bar{4}_5\ \bar{2}_5$	5778, 9349, 3571 (L_{18}, L_{19}, L_{17})	$\bar{3}_5\ \bar{4}_5\ \bar{1}_5$
no	41	165580141 267914296 102334155	$\bar{1}_5\ \bar{1}_5\ \bar{0}_5$	24476, 39603, 15127 (L_{21}, L_{22}, L_{20})	$\bar{1}_5\ \bar{3}_5\ \bar{2}_5$
yes	43	433494437 701408733 267914290	$\bar{2}_5\ \bar{3}_5\ \bar{2}_5$	10946, 17711, 6765 (F_{21}, F_{22}, F_{20})	$\bar{1}_5\ \bar{1}_5\ \bar{0}_5$
?	47	2971215073 4807526976 1836311903	$\bar{3}_5\ \bar{1}_5\ \bar{3}_5$	28657, 46368, 17711 (F_{23}, F_{24}, F_{22})	$\bar{2}_5\ \bar{3}_5\ \bar{1}_5$
?	53	53316291173 86267571272 32951280099	$\bar{3}_5\ \bar{2}_5\ \bar{4}_5$	271443, 439204, 167761 (L_{26}, L_{27}, L_{25})	$\bar{3}_5\ \bar{4}_5\ \bar{1}_5$
?	59	956722026041 1548008755920 591286729879	$\bar{1}_5\ \bar{0}_5\ \bar{4}_5$	832040, 1346269, 514229 (F_{30}, F_{31}, F_{29})	$\bar{0}_5\ \bar{4}_5\ \bar{4}_5$
?	61	2504730781961 4052739537881 1548008755920	$\bar{1}_5\ \bar{1}_5\ \bar{0}_5$	3010349, 2178309, 832040 (L_{31}, L_{32}, L_{30})	$\bar{4}_5\ \bar{4}_5\ \bar{0}_5$
?	67	44945570212853 72723460248141 27777890035288	$\bar{3}_5\ \bar{1}_5\ \bar{3}_5$	3524578, 5702887, 2178309 (F_{33}, F_{34}, F_{32})	$\bar{3}_5\ \bar{2}_5\ \bar{4}_5$

Table 4. Class structure of primitive Fibonacci triples

3 Pascal's triangle and Fibonacci numbers

It is well known that the Fibonacci numbers can be generalised by summing along the leading diagonals in Pascal's triangle. That is,

$$F_n = \sum_{j=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n-j-1}{j}.$$

When $n = p$, prime this can be conveniently re-written as

$$F_p = 2 + \sum_{j=2}^{\left\lfloor \frac{pn+1}{2} \right\rfloor} \binom{p-j}{j-1}. \quad (3.1)$$

We shall call these numbers Pascal-Fibonacci numbers (Table 5). An equivalent sum to that in (3.1) is

$$\sum_{i=2}^{\left\lfloor \frac{p-1}{2} \right\rfloor} \binom{p-i}{i-1},$$

so that for each Pascal-Fibonacci number, $N_{PF}(i-1)$, along each diagonal is given by

$$N_{PF} = \binom{p-i}{i-1}.$$

For example, when p = 17 and i = 4, the third number in the sum is $N_{P_{17}}(3) = 286$. Similarly, when p = 43 and i = 5, $N_{P_{43}}(4) = 73815$. Again, when p = 59, the last i = ½(p – 1) =29, so the 28th number in the sum is $N_{P_{59}}(28) = 30!/28!\times 2! = 435$.

p	F_p
7	5, 6
11	9, 28, 35, 15
13	11, 45, 84, 70, 21
17	15, 91, 286, 495, 462, 210, 36
19	17, 120, 455, 1001, 1287, 924, 330, 45
23	21, 190, 969, 3060, 6188, 8008, 6435, 3003, 715, 66
29	27, 325, 2300, 10626, 33649, 74613, 116280, 125970, 92378, 43758, 12376, 1820, 105
31	29, 378, 2925, 14950, 53130, 134596, 245157, 319770, 293930, 184756, 75582, 18564, 2380, 120
37	35, 561, 5456, 35960, 169911, 593775, 1560780, 3108105, 4686825, 5311735, 4457400, 2704156, 1144066, 319770, 54264, 48451, 171
41	39, 703, 7770, 58905, 324632, 1344904, 4272048, 10518300, 20160075, 30045015, 34597290, 30421755, 20058300, 9657700, 3268760, 735471, 100947, 7315, 210
43	41, 780, 9139, 73815, 435897, 1947792, 6724520, 18156204, 38567100, 64512240, 84672315, 86493225, 67863915, 40116600, 17383860, 5311735, 1081575, 134596, 8855, 231
47	45, 946, 12341, 111930, 749398, 3838380, 15380937, 48903492, 124403620, 254186856, 417225900, 548354040, 573166440, 471435600, 300540195, 145422675, 51895935, 13123110, 2220075, 230230, 12650, 276
53	51, 1225, 18424, 194580, 1533939, 9366819, 45379620, 177232627, 563921995, 1471442973, 3159461968, 5586853480, 8122425444, 9669554100, 9364199760, 7307872110, 4537567650, 2203961430, 818809200, 225792840, 44352165, 5852925, 475020, 20475, 351
59	57, 1540, 26235, 316251, 2869685, 20358520, 115775100, 536878650, 2054455634, 6540715896, 17417133617, 38910617655, 73006209045, 114955808528, 151532656696, 166509721602, 166509721602, 151584480450, 113380261800, 68923264410, 33578000610, 12875774670, 3796297200, 8344518000, 131128140, 13884156, 906192, 34165, 435

Table 5. Pascal-Fibonacci numbers

As can be seen from above in the distinction between p^* = 3, 7 and p^* = 1, 9, the class of p in Z_5 given by the REDs is critical to the structure. Therefore, we compare the Z_5 structure of the Pascal-Fibonacci numbers on this basis (Table 6). For example, for p^* = 1, the first numbers 9, 29, 39, 59 equal $(5r_4+4) \in \bar{4}_5$, the second numbers fall in class $\bar{3}_5(5r_3+3)$, and the third and fourth fall in $\bar{0}_5\,(5r_0)$.

The proportion of the Pascal-Fibonacci numbers in the various categories (Table 7) shows trends that are characteristic of primality. For example, the Pascal-Fibonacci composite numbers show lack of balance in the distribution of parity. The ‘dominance’ of the Class $\bar{0}_5$ is common to all the Pascal-Fibonacci numbers. This latter characteristic is emphasised particularly in the case of the non-prime F_p values.

If various Pascal-Fibonacci numbers are added and then checked for a common factor, this could indicate primality. For instance, for $p = 19$: $(17 + 120 + 455) = 592 = 37 \times 16$. Addition of remaining numbers in sum plus 2 yields $3589 = 37 \times 97$, so that F_{19} is composite and equals 37×113.

p	**Pascal-Fibonacci Integers in Z_5**									
11	$\bar{4}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{0}_5$						
31	$\bar{4}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{1}_5$
41	$\bar{4}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{2}_5$	$\bar{4}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$
61	$\bar{4}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{1}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$
13	$\bar{1}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{0}_5$	$\bar{1}_5$					
23	$\bar{1}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{0}_5$	$\bar{3}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{1}_5$
43	$\bar{1}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{0}_5$	$\bar{2}_5$	$\bar{2}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{0}_5$	$\bar{0}_5$
53	$\bar{1}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{4}_5$	$\bar{0}_5$	$\bar{2}_5$	$\bar{0}_5$	$\bar{3}_5$
7	$\bar{0}_5$	$\bar{1}_5$								
17	$\bar{0}_5$	$\bar{1}_5$	$\bar{1}_5$	$\bar{0}_5$	$\bar{2}_5$	$\bar{0}_5$	$\bar{1}_5$			
37	$\bar{0}_5$	$\bar{1}_5$	$\bar{1}_5$	$\bar{0}_5$	$\bar{1}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$
47	$\bar{0}_5$	$\bar{1}_5$	$\bar{1}_5$	$\bar{0}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{2}_5$	$\bar{2}_5$	$\bar{2}_5$	$\bar{1}_5$
19	$\bar{2}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{4}_5$	$\bar{0}_5$	$\bar{0}_5$		
29	$\bar{2}_5$	$\bar{0}_5$	$\bar{1}_5$	$\bar{4}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{3}_5$	$\bar{3}_5$	$\bar{1}_5$
59	$\bar{2}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{1}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{1}_5$

Table 6. Pascal-Fibonacci integers in Z_5

p	**even**	**odd**	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{4}_5$
7	50	50	50	50	0	0	0
11	25	75	50	0	0	0	50
13	40	60	40	40	0	0	20
17	57	43	43	43	14	0	0
19	37	63	50	12.5	25	0	12.5
23	50	50	40	20	0	30	10
29	62	38	46	15	8	23	8
31	79	21	51	14	14	7	14
37	47	53	53	41	0	0	6
41	53	47	63	5.5	10.5	10.5	10.5
43	45	55	60	15	15	0	10
47	67	33	67	19	9	5	0
53	52	48	56	8	8	8	20
59	70	30	66	14	17	10	3

Table 7. Proportions of Fibonacci numbers in various classes (modulo 5)

4 F_p as a function of F_{p+1}

(i) $p^* = 1, 9$

The Fibonacci triples for prime p are related by Simson's identity [2] which may be stated in the form

$$F_p^2 = F_{p+1}F_{p-1} + 1 \tag{4.1}$$

and from Tables 2, 3, they can be expressed as

$$F_p = k_1 p + 1, \quad F_{p-1} = k_2 p, \quad F_{p+1} = k_3 p + 1.$$

For $k_1, k_2, k_3 \in \boldsymbol{Z}$. From this we get

$$F_p = AF_{p+1} - 1 \tag{4.2}$$

in which

$$A = \frac{k_2}{k_1} = \frac{F_{p-1}}{F_p - 1}$$

A comparison of A for various values of p (Table 8a) shows that generally for primes

$$A = \frac{F_m + F_n}{F_{m+2} - F_n}$$

where $m = ½(p + 1)$ and $n = m – 2$, whereas non-primes have a pattern

$$A = \frac{F_m}{F_n}$$

where $m = ½(p – 1)$ and $n = ½(p + 1)$.

p	***A* as a** Numerical fraction	Fibonacci or Lucas ratio	F_{p+1}	F_p
11	5/8	F_5/F_6	144	89
19	2×17/5×11	F_9/F_{10}	6765	4181
29	3×281/4×11×31	L_{14}/L_{15}	832040	514229
31	2×5×61/3×7×47	F_{15}/F_{16}	2178309	1346269
41	7×2161/4×29×211	L_{20}/L_{21}	267914296	165580141
59	514229/5×8×11×31×61	F_{29}/F_{30}	1548008755920	956722026041
61	2×3×3×41×2521/3010349	L_{30}/L_{31}	4052739537881	2504730781961

Table 8(a). Equation (4.2) A for p^* =1, 9

(ii) $p^* = 3, 7$

From Tables 2, 3, we can get

$$F_p = k_1 p - 1, \quad F_{p-1} = k_2 p + 1, \quad F_{p+1} = k_3 p.$$

For $k_1, k_2, k_3 \in \boldsymbol{Z}$. From this we get

$$F_p = AF_{p+1} + 1 \tag{4.3}$$

in which

$$A = \frac{k_2 p + 1}{k_1 p} = \frac{F_{p-1}}{F_p + 1}$$

As in (i), primes have

$$A = \frac{F_m + F_n}{F_{m+2} - F_n}$$

where $m = ½(p + 1)$ and $n = m – 2$, and primes also have a pattern

$$A = \frac{F_m}{F_{m+1}}$$

where $m = ½(p − 1)$. A comparison of A for various values of p is displayed below (Table 7b).

p	***A*** **as a numerical fraction**	**Fibonacci or Lucas ratio**	F_{p+1}	F_p
3	1/3	L_1/L_2	3	2
7	4/7	L_3/L_4	21	13
13	8/13	F_6/F_7	377	233
17	21/34	F_8/F_9	2584	1597
23	199/14×23	L_{11}/L_{12}	46368	28657
37	8×17×19/113×37	F_{18}/F_{19}	39088169	24157817
43	4×29×211/3×43×307	L_{21}/L_{22}	701408733	433494437
47	461×139/2×47×1103	L_{23}/L_{24}	4807526976	2971215073
53	521×233/2×53×1853	F_{26}/F_{27}	86267571272	53316291173

Table 8(b). Equation (4.2) A for $p^* = 3, 7$

In contrast to the primitive triples for $p^* = 3, 7$, the last two digits of p are (even, odd) for Lucas and (odd, odd) for Fibonacci. For $p^* = 1, 9$, the last two digits linked to Lucas and Fibonacci are the same as the primitive triple one.

5 Fibonacci squares

(a) F_{2p+1}

When we combine Simson's identity with the equally well-known [7]

$$F_n^2 + F_{n+1}^2 = F_{2n+1} \tag{5.1}$$

we get for $n = p$ (odd):

$$F_{p+1}F_{p-1} + F_{p+1}^2 = F_{2p+1} - 1 \tag{5.2}$$

When $p^* \in \{3, 7\}$, $p \mid F_{p+1}$ (Table 2), and so from (5.2) $p \mid (F_{2p+1} – 1)$ (Table 9a).

p	**2p + 1**	$F_{2p+1} - 1$		
13	27	196417	=13×15109	=13(29×521)
17	35	9227464	=17×542792	=17(2^3×19x3571)

23	47	2971215072	=23×129183264	= 23(2^5×3^3×7×139×461)
37	75	2111485077978649	= 37×57067164269677	= 37(57067164269677)

Table 9(a). $p \mid (F_{2p+1} - 1)$

Obviously, F_{2p+1} may be obtained directly from (5.2) so that when $2p$+1 is prime this gives additional information on the production of primes. Note that for this set $F_{2p+1}-1 = pQ$, where $Q \in \bar{4}_5$ for $p^* = 3$, and $Q \in \bar{2}_5$ for $p^* = 7$.

When $p^* \in \{1, 9\}$, $p \mid F_{p-1}$ (Table 2), and so from (5.1) and (5.2) $p \mid (F_{2p-3} - 1)$ (Table 9b).

Note that for this set $(F_{2p-3} - 1) = pQ$, where Q is even and $\in \bar{0}_5$ for $p^* = 1$, and $Q \in 1_5$ for $p^* = 9$.

p	**2p – 3**	**$F_{2p-3} - 1$**		
11	19	4180	= 11×380	= 11(2^2×5×19))
19	35	9227464	= 19×485656	= 19(2^3×17×3571)
29	55	139583862444	= 29×4813236636	= 29(2^2×3×13×19×281×5779)
31	59	956722026040	= 31×30862000840	= 31(2^3×5×8×11×59×19489)

Table 9(b). $p \mid (F_{2p-3} - 1)$

(b) Class patterns of p^2 and F_p^2

Since $F_p^2{}^* = p^2 * F_p^2$ and p^2 are in the same class, then $\left(F_p^2 - p^2\right) \in \bar{0}_5$ (Table 10). For $p^* \in \{3, 7\}$, $F_p^2 - p^2$ has 120 as the last three digits (Classes $\bar{2}_5$, $\bar{3}_5$), whereas for $p^* \in \{1, 9\}$, $F_p^2 - p^2$ has X00 as the last three digits (Classes $\bar{1}_5$, $\bar{4}_5$). For composites only $X = 4$ in the range considered. This difference in structure might be another indication of primality. (For $p = 41$, 43, $F_p^2 - p^2$ ends in 8200 and 5120, respectively.)

p	**F_p**	F_p^2	**p^2**	$F_p^2 - p^2$
5	5	25	25	0
7	13	169	49	120
11	89	7921	121	7800
13	233	54289	169	54120
17	1597	2550409	289	2550120
19	4181	17480761	361	17480400
23	28657	821223649	529	821223120
29	514229	264431464441	841	264431463600
31	1346269	1812440220361	961	1812440219400
37	24157817	582413122205489	1369	582413122204120

Table 10. $F_p^2 - p^2$

6 The influence of F_{p+2}, F_{p+3}

Since [7], p is odd,

$$F_{p-2}F_{p+2} = F_p^2 + 1 \tag{6.1}$$

and

$$F_{p-3}F_{p+3} = F_p^2 - 4 \tag{6.2}$$

so that from Simson's identity and (6.1), we get

$$F_{p-1}F_{p+1} = F_{p-2}F_{p+2} - 2. \tag{6.3}$$

Thus $p \mid \left(F_{p-2}F_{p+2} - 2\right)$ (Table 11), in which 24 is always a factor of $\left(F_{p-2}F_{p+2} - 2\right)$.

p	F_p	F_{p-2}	F_{p+2}	$((F_{p-2} \times F_{p+2}) - 2)/p$
7	13	5	34	24
11	89	34	233	24(2×15)
13	233	89	610	24(2×87)
17	1597	610	4181	24(7×893)
19	4181	1597	10946	24(5×11×697)
23	28657	10946	75025	24(2^2×3×7×89×199)
29	514229	196418	1346269	24(5×11×13×31×61×281)
31	1346269	514229	3524578	24(5×19×331×77471)
37	24157817	9227465	39088169	24(2×3×17×19×339116277)

Table 11: $((F_{p-2} \times F_{p+2}) - 2) / p$

The last remark is a consequence of the opposite parity of F_{p-1} and F_{p+1} with 3 as a factor of either and 8 as a factor of the even number (Table 12).

p	F_{p+1}	Factors of		F_{p-1}
		F_{p+1}	F_{p-1}	
7	21	3, X	8	8
11	144	3, 8	X	55
13	377	X	3, 8	144
17	2584	8, X	3	987
19	6765	3	8, X	2584
23	46368	3, 8, X	–	17711
29	832040	8	3, X	317811
31	2178309	3	8, X	832040
37	39088169	X	3, 8	14930352
41	267914296	8	3, X	102334155
43	701408733	3, X	8	267914296
47	4807526976	X	3, 8	1836311903
53	86267571272	8, X	3	32951280099
59	1548008755920	3, 8	X	591286729879
61	4052739537881	–	3, 8, X	1548008755920

Table 12. 'X' divisible by p

The distribution of 3 and 8 suggests that some composites have F_{p-1} even and divisible by 8 as well as p. Many cases have the factors 3 and 8 confined to one of the numbers adjacent to F_p. However, p^* effects are overlaid.

Comparison of the factors of F_{p-2} and F_{p-1} shows that none are common.In fact, for the composites F_{19} and F_{31}, F_{p-2} values are prime. However, the prime F_{13} also has this feature. Comparison of F_{p-2} and F_{p-3} yields the same result.

7 Concluding comments

The variety of structural patterns presented here show that differences exist between Fibonacci prime-suffixed index numbers, F_p, which produce prime numbers, and those which produce composite numbers. While the overall structure of F_p is stable, as can be inferred from Equations (1.1) and (3.1), there is no reason, at least from a structural point of view, why there should not be an infinity of Fibonacci primes.

Further analysis using different modular rings should be useful including the more general Z_p, considered as the ring of p-adic generalized integers [1, 4].

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