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Method of infinite ascent applied on $A^3 \pm nB^2 = C^3$

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Abstract: In this paper we will produce different formulae for which the Diophantine equation $A^3 \pm nB^2 = C^3$ will generate infinite number of co-prime integral solutions for $(A\,, B\,, C)$ for any positive integer n.
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1 Introduction

The study of Diophantine equations and to find their solutions would continue to puzzle both mathematicians and amateurs alike. In a series of papers [4, 5, 6, 7] Jena has used the Method of Infinite Ascent to find infinite number of co-prime integral solutions for the parameters of different Diophantine equations. To apply this method to any Diophantine problem, we need to discover the appropriate algebraic identity linked with the Diophantine equation under consideration. At present, there seems to be no simple way to find these algebraic identities. The integral co-prime solutions of the Diophantine equations as given in (1) are yet to be studied for all positive integral values of n. Of course, for $n = 1$, these equations have been thoroughly studied by Beuker [1], who gives a complete list of parametric solutions of the Diophantine equation $A^3 + B^3 = C^2$. In a paper [8] Kraus studied the Diophantine equation $a^3 + b^3 = c^p$ to show the impossibility of primitive integral solutions for $a\,, b$ and c for $17 \leq p \leq 10000$, where p is a prime number. Bruin [2, 3] solved this equation for p = 4 and 5. But, in this paper, we study the two families of Diophantine equations of (1) relating to all positive integral values of n. Let us consider the following Diophantine equations

$$A^3 \pm nB^2 = C^3 \tag{1}$$

[1]Thankful to my parents - my heavenly mother and revered father, for allowing me to dream!

2 The Diophantine equations $A^3 \pm nB^2 = C^3$

Theorem 1 and Theorem 2 suggest that each of the two Diophantine equations of (1) has infinite number of co-prime integral solutions for $(A\,, B\,, C)$ for any positive integer n. But, before we proceed further, let us state and prove the following Lemmas which would be used in proving these theorems.

Lemma 1. *For any two positive integers* p *and* q*,*

$$(3p^2+6pq-q^2)^2+(3p^2+6pq-q^2)(3p^2-6pq-q^2) + (3p^2-6pq-q^2)^2 = 3(3p^2+q^2)^2 \tag{2}$$

Proof. Expanding the three constituent terms in the LHS of (2), we get

$$\begin{aligned}(3p^2 &+ 6pq-q^2)^2\\ &= (3p^2)^2+(6pq)^2+(-q^2)^2+2(3p^2)(6pq)+2(6pq)(-q^2)+2(-q^2)(3p^2)\\ &= 9p^4+36p^2q^2+q^4+36p^3q-12pq^3-6p^2q^2\\ &= 9p^4+36p^3q+(36-6)p^2q^2-12pq^3+q^4\\ &= 9p^4+36p^3q+30p^2q^2-12pq^3+q^4\end{aligned} \tag{3}$$

$$\begin{aligned}(3p^2&+ 6pq-q^2)(3p^2-6pq-q^2)\\ &= 3p^2(3p^2-6pq-q^2)+6pq(3p^2-6pq-q^2)-q^2(3p^2-6pq-q^2)\\ &= 9p^4-18p^3q-3p^2q^2+18p^3q-36p^2q^2-6pq^3-3p^2q^2+6pq^3+q^4\\ &= 9p^4+(-18+18)p^3q+(-3-36-3)p^2q^2+(-6+6)pq^3+q^4\\ &= 9p^4-42p^2q^2+q^4\end{aligned} \tag{4}$$

$$\begin{aligned}(3p^2 &- 6pq-q^2)^2\\ &= (3p^2)^2+(-6pq)^2+(-q^2)^2+2(3p^2)(-6pq)+2(-6pq)(-q^2)+2(-q^2)(3p^2)\\ &= 9p^4+36p^2q^2+q^4-36p^3q+12pq^3-6p^2q^2\\ &= 9p^4-36p^3q+(36-6)p^2q^2+12pq^3+q^4\\ &= 9p^4-36p^3q+30p^2q^2+12pq^3+q^4\end{aligned} \tag{5}$$

L. H. S. of (2) [From (3), (4) and (5)]

$$\begin{aligned}&= (3p^2+6pq-q^2)^2+(3p^2+6pq-q^2)(3p^2-6pq-q^2)+(3p^2-6pq-q^2)^2\\ &= (9p^4+36p^3q+30p^2q^2-12pq^3+q^4)+(9p^4-42p^2q^2+q^4)\\ &\qquad +(9p^4-36p^3q+30p^2q^2+12pq^3+q^4)\\ &= (9+9+9)p^4+(36-36)p^3q+(30-42+30)p^2q^2\\ &\qquad +(-12+12)pq^3+(1+1+1)q^4\\ &= 27p^4+18p^2q^2+3q^4 = 3(9p^4+6p^2q^2+q^4) = 3\{(3p^2)^2+2.(3p^2)(q^2)+(q^2)^2\}\\ &= 3(3p^2+q^2)^2 = \text{R. H. S. of (2).}\end{aligned}$$

Hence, Lemma 1 is proved. □

Lemma 2. *For any two positive integers p and q,*

$$(3p^2 - 6pq - q^2)^3 + 36pq(3p^2 + q^2)^2 = (3p^2 + 6pq - q^2)^3 \tag{6}$$

Proof. Now, $(3p^2 + 6pq - q^2)^3 - (3p^2 - 6pq - q^2)^3$

$$\begin{aligned}
&= \{(3p^2 + 6pq - q^2) - (3p^2 - 6pq - q^2)\}\times \\
&\quad \{(3p^2 + 6pq - q^2)^2 + (3p^2 + 6pq - q^2)(3p^2 - 6pq - q^2) + (3p^2 - 6pq - q^2)^2\} \\
&= \{(3p^2 + 6pq - q^2 - 3p^2 + 6pq + q^2)\} \cdot \{3(3p^2 + q^2)^2\} \text{ [From (2)]} \\
&= 12pq \cdot 3(3p^2 + q^2)^2 = 36pq(3p^2 + q^2)^2
\end{aligned}$$

$$\begin{aligned}
&\text{So}\,, (3p^2 + 6pq - q^2)^3 - (3p^2 - 6pq - q^2)^3 = 36pq(3p^2 + q^2)^2 \\
&\Longrightarrow (3p^2 + 6pq - q^2)^3 = (3p^2 - 6pq - q^2)^3 + 36pq(3p^2 + q^2)^2 \\
&\Longrightarrow (3p^2 - 6pq - q^2)^3 + 36pq(3p^2 + q^2)^2 = (3p^2 + 6pq - q^2)^3.
\end{aligned}$$

Hence, Lemma 2 is proved. □

Lemma 3. *For any two positive integers p and q,*

$$(3p^2 + 6pq - q^2)^3 - 36pq(3p^2 + q^2)^2 = (3p^2 - 6pq - q^2)^3$$

Proof. From (6) we get,

$$\begin{aligned}
&(3p^2 + 6pq - q^2)^3 = (3p^2 - 6pq - q^2)^3 + 36pq(3p^2 + q^2)^2 \\
\Longrightarrow\ &(3p^2 + 6pq - q^2)^3 - 36pq(3p^2 + q^2)^2 = (3p^2 - 6pq - q^2)^3
\end{aligned}$$

Hence, Lemma 3 is proved. □

Theorem 1. *For any positive integer n, the Diophantine equation $A^3 + nB^2 = C^3$ has infinitely many co-prime integral solutions for*

$$(A\,, B\,, C) = \{(m^4 - 6m^2n - 3n^2)\,, 6m(m^4 + 3n^2)\,, (m^4 + 6m^2n - 3n^2)\}$$

where $m \neq 0$ and can take any integral value not divisible by 3 so that m, n are co-prime and one is odd, the other is even.

Proof. We will prove Theorem 1 in two steps.

Step I. According to the statement of the Theorem 1, if

$$A^3 + nB^2 = C^3, \tag{7}$$

then, we have to establish that

$$(m^4 - 6m^2n - 3n^2)^3 + n\{6m(m^4 + 3n^2)\}^2 = (m^4 + 6m^2n - 3n^2)^3 \tag{8}$$

Now, in (6), substitute $p = m^2$ and $q = 3n$
Hence,

$$(3m^4 - 6m^2.3n - 9n^2)^3 + 36m^2.3n(3m^4 + 9n^2)^2 = (3m^4 + 6m^2.3n - 9n^2)^3$$
$$\implies 3^3(m^4 - 6m^2n - 3n^2)^3 + 36m^2 \cdot 3^3n(m^4 + 3n^2)^2 = 3^3(m^4 + 6m^2n - 3n^2)^3$$
$$\implies (m^4 - 6m^2n - 3n^2)^3 + 36m^2n(m^4 + 3n^2)^2 = (m^4 + 6m^2n - 3n^2)^3$$
$$\implies (m^4 - 6m^2n - 3n^2)^3 + n\{6m(m^4 + 3n^2)\}^2 = (m^4 + 6m^2n - 3n^2)^3$$

which establishes (8). Alternatively, in (6), substitute $p = n$ and $q = m^2$, and with a little manipulation we can get the algebraic identity as given by (8).

Step II. We have to show that the given integral solutions of (7) for $(A\,, B\,, C)$ to be pair-wise co-prime.
Now. $A = (m^4 - 6m^2n - 3n^2)$, which would be odd, because m and n are of opposite parity. $B = 6m(m^4 + 3n^2)$ and it is always even; and $C = (m^4 + 6m^2n - 3n^2)$, which would be odd, because m and n are of opposite parity. Since, 3 is not a factor of m, both A and C are not divisible by 3. Again, $(m^4 - 3n^2)$ which is odd, and $6m^2n$ will not have a common factor $k > 1$. So, A and C would be co-prime because, A is the difference, and C is the sum of two co-prime terms $(m^4 - 3n^2)$ and $6m^2n$. Knowing that A and C are co-prime, from equation (7) we see that $(A\,, B\,, C)$ would be pair-wise co-prime.

Thus, combing Step I and Step II, Theorem 1 is completely established. □

Theorem 2. *For any positive integer n, the Diophantine equation $A^3 - nB^2 = C^3$ has infinitely many co-prime integral solutions for*

$$(A\,, B\,, C) = \{(m^4 + 6m^2n - 3n^2)\,, 6m(m^4 + 3n^2)\,, (m^4 - 6m^2n - 3n^2)\}$$

where $m \neq 0$ and can take any integral value not divisible by 3 so that m, n are co-prime and one is odd, the other is even.

Proof. If we change n to $-n$ in Theorem 1, the statement of Theorem 2 follows. So, in proving this Theorem, we have to follow the arguments of proof in Theorem 1 after replacing n with $-n$. □

3 Comments

The present paper is not an attempt to find the complete set of parametric solutions to the two families of Diophantine equations (1). Hence, there is a scope of further research on these equations.

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References

[1] Beukers, F. The Diophantine equation $Ax^p + By^q = Cz^r$, *Duke Math. J.*, Vol. 91, 1998, 61–88.

[2] Bruin, N. On powers as sums of two cubes, **ANTS IV**, Leiden, 2000, *Lecture Notes in Comp. Sci.*, Springer, Vol. 1838, 2000, 169–184.

[3] Bruin, N. Chabauty methods using elliptic curves, *J. reine angew. Math.*, Vol. 562, 2003, 27–49.

[4] Jena, S. K. Method of Infinite Ascent applied on $mA^6 + nB^3 = C^2$, *The Math. Stud.*, Vol. 77, 2008, No. 1–4, 239–246.

[5] Jena, S. K. Method of Infinite Ascent applied on $A^4 \pm nB^2 = C^3$, *The Math. Stud.*, Vol. 78, 2009, No. 1–4, 233–238.

[6] Jena, S. K. Method of Infinite Ascent applied on $mA^3 + nB^3 = C^2$, *The Math. Stud.*, Vol. 79, 2010, No. 1–4, 187–192.

[7] Jena, S. K. Method of Infinite Ascent applied on $mA^3 + nB^3 = 3C^2$, *The Math. Stud.*, Vol. 81, 2012, No. 1–4 (To appear).

[8] Kraus, A. Sur $l'equation\ a^3 + b^3 = c^p$, *Experiment Math. J.*, Vol. 71, 1998, No. 1, 1–13.