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On new refinements of Kober's and Jordan's trigonometric inequalities

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Abstract: This paper deals with some inequalities for trigonometric functions such as the Jordan inequality and Kober's inequality with refinements. In particular, lower and upper bounds for functions such as $(\sin x)/x$, $(1-\cos x)/x$ and $(\tan x/2)/x$ are proved.
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1 Introduction

During the past several years there has been a great interest in trigonometric inequalities (see e.g. a list of papers in [2], where 46 items are included).

The classical Jordan inequality

$$\frac{2}{\pi}x \le \sin x, \quad 0 \le x \le \frac{\pi}{2} \tag{1}$$

has been in the focus of these studies and many refinements have been proved (see e.g. the references from [2], [12], [13], [4]). One of the early references from this topic is the author's book [7] on Geometric inequalities (with a geometric interpretation of (1), too, which was republished also in the book [16]) or the author's papers from 2001 ([9]) and 2005 ([10], [11]), or more recently, from 2007 ([13]). See also the book [15]. For example, in [9] it was proved that

$$\frac{1+\cos x}{2} < \frac{\sin x}{x} < \cos\frac{x}{2}, \quad 0 < x < \frac{\pi}{2}, \tag{2}$$

rediscovered, and used many times in the literature (see e.g. [2], relations (1.2) and (1.10)). For an application of Jordan's inequality in Number theory, see e.g. [17].

Another famous inequality, connected with Jordan's inequality is Kober's inequality (see e.g. [3])

$$\cos x \geq 1 - \frac{2}{\pi}x, \quad 0 \leq x \leq \frac{\pi}{2}. \tag{3}$$

Though (3) can be proved by considering the monotonic property of the function $\frac{1-\cos x}{x}$, in [10], [11], [13] we have remarked that, it follows at once via the substitution $x \to \frac{\pi}{2} - x$ in relation (1); and vice-versa, (3) implies (1) by the same manner. In paper [11] we have shown that the application $g(x) = \sin x/x$, $0 < x \leq \frac{\pi}{2}$, $g(0) = 1$, is a strictly increasing and strictly concave function on $\left[0, \frac{\pi}{2}\right]$. See also [14]. By writing that the graph of line passing on the points $A(0,1)$ and $B\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$ from the graph of g is below the graph of g, one obtains that

$$\frac{\sin x}{x} \geq 1 + \frac{2(2-\pi)}{\pi^2} \cdot x, \tag{4}$$

which is a refinement of (1), as the right side of (4) may be written also as

$$\frac{2}{\pi} + \frac{\pi - 2}{\pi^2}(\pi - 2x) \geq \frac{2}{\pi}.$$

By writing the tangent line to the graph of function g at the point B, by the concavity of g one gets

$$\frac{\sin x}{x} \leq \frac{4}{\pi} - \frac{4}{\pi^2} \cdot x, \tag{5}$$

which is a counterpart of (4) (see [13] for details). Such inequalities will be used in the next sections.

2 A counterpart of Jordan's inequality

Another inequality, which may be found also in [13] (see relation (18)) is the following:

$$\frac{\tan \frac{x}{2}}{x} \leq \frac{2}{\pi}, \quad 0 \leq x \leq \frac{\pi}{2}. \tag{6}$$

Here the left side is interpreted in $x = 0$ as $\lim\limits_{x \to 0} \frac{\tan \frac{x}{2}}{x} = \frac{1}{2}$.

For a new proof of relation (6), let us remark that, the graph of convex function $x \to \tan \frac{x}{2}$ $\left(0 \leq x \leq \frac{\pi}{2}\right)$ is below the segment line passing through the points $(0,0)$ and $\left(\frac{\pi}{2}, 1\right)$.

In what follows, (6) will be called as "a counterpart of Jordan's inequality".

Our first result shows that this inequality refines Kober's inequality:

Theorem 1. *One has*

$$\frac{1-\cos x}{x} \leq \frac{\tan \frac{x}{2}}{x} \leq \frac{2}{\pi}, \quad 0 < x \leq \frac{\pi}{2}. \tag{7}$$

Proof. By $1 - \cos x = 2\sin^2 \frac{x}{2}$ and $2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x$, the inequality $1 - \cos x \le \tan\frac{x}{2}$ becomes $\sin x \le 1$, which is true, with equality only for $x = \frac{\pi}{2}$.

Another proof can be obtained by letting $\tan\frac{x}{2} = t$, and using the formula $\cos x = \frac{1-t^2}{1+t^2}$. Then we have to prove that,

$$\frac{2t^2}{1+t^2} \le t \quad \text{or} \quad 2t \le 1 + t^2,$$

which is $(t-1)^2 \ge 0$, etc. $\square$

Now we will obtain a lower bound for $\left(\tan\frac{x}{2}\right)/x$:

Theorem 2.

$$\frac{1}{\pi - x} \le \frac{\tan\frac{x}{2}}{x} \le \frac{2}{\pi}, \quad 0 \le x \le \frac{\pi}{2}. \tag{8}$$

Proof. We will obtain a method of proof of (3), suggested in Section 1. Put $\frac{\pi}{2} - x$ in place of x in inequality (6). As

$$\tan\left(\left(\frac{\pi}{2} - \frac{x}{2}\right)/2\right) = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}},$$

after some elementary transformations we get the left side of (8). $\square$

Remark 1. It can be proved immediately that, the lower bounds of (7), resp. (8) cannot be compared, i.e. some of the inequalities

$$\frac{1 - \cos x}{x} < \frac{1}{\pi - x} \quad \text{and} \quad \frac{1 - \cos x}{x} > \frac{1}{\pi - x}$$

is true for all $x \in \left(0, \frac{\pi}{2}\right)$.

Strong improvements of the left side of (8) will be obtained by another methods (see Theorem 3).

Though $\frac{1-\cos x}{x}$ and $\frac{1-\cos x}{x^2}$ cannot be compared for all $x \in \left(0, \frac{\pi}{2}\right)$ (only for $0 < x \le 1$ or $1 \le x < \frac{\pi}{2}$), the later one is also a lower bound for $(\tan x/2)/x$. More generally, the following inequalities are true.

Theorem 3.

$$\frac{1}{\pi} \le \frac{\sin x}{2x} \le \frac{1 - \cos x}{x^2} \le \frac{\sin\frac{x}{2}}{x} \le \frac{1}{2} \le \frac{\tan\frac{x}{2}}{x} \le \frac{2}{\pi} \tag{9}$$

Proof. The first inequality of (9) is exactly Jordan's inequality (1). Applying

$$1 - \cos x = 2\sin^2\frac{x}{2} \quad \text{and} \quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2},$$

the second inequality follows by $\frac{x}{2} \leq \tan\frac{x}{2}$, while the third one, by

$$\sin\frac{x}{2} \leq \frac{x}{2}.$$

The lower bound $\frac{1}{\pi}$ however, is not the best one for

$$q(x) = \frac{1-\cos x}{x^2}, \quad 0 < x < \frac{\pi}{2}.$$

□

Lemma 1. *The function q defined above is strictly decreasing and strictly concave.*

Proof. After some elementary computations (which we omit here) one obtains

$$q'(x) = \frac{x\sin x + 2\cos x - 2}{x^3},$$

$$x^4 q''(x) = x^2\cos x - 4x\sin x - 6\cos x + 6 = p(x),$$

$$p'(x) = 2\sin x - 2x\cos x - x^2\sin x, \quad p''(x) = -x^2\cos x < 0.$$

As $p''(x) < 0$, one gets $p'(x) < p'(0) = 0$ for $x > 0$, so $p(x) < p(0) = 0$, which shows that $q''(x) < 0$, i.e. q is strictly concave.

By letting $r(x) = x\sin x + 2\cos x - 2$, remark that $r(x) < 0$ can be written also as

$$\frac{\sin x}{2x} \leq \frac{1-\cos x}{x^2},$$

which is the second inequality of (9). Thus q is strictly decreasing. □

Theorem 4.

$$\frac{4}{\pi^2} \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2} \leq \frac{\tan\frac{x}{2}}{x} \leq \frac{2}{\pi}; \tag{10}$$

$$\frac{4}{\pi^2} \leq \frac{1-\cos x}{x^2} \leq \frac{4}{\pi^2} + \frac{4(4-\pi)}{\pi^3}\left(\frac{\pi}{2} - x\right); \tag{11}$$

$$\frac{1}{2} - \frac{\pi^2-8}{\pi^3}x \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2}. \tag{12}$$

Proof. The first two inequalities of (10) are consequences of

$$q\left(\frac{\pi}{2}\right) \leq q(x) \leq q(0+) = \lim_{x\to 0} q(x) = \frac{1}{2}.$$

Now, we have essentially to prove the right side of (11), as well as the left side of (12).

We will use the method of proof of inequality (5). By writing the tangent line to the graph of function q at the point $B\left(\frac{\pi}{2}, \frac{4}{\pi^2}\right)$, as $q'\left(\frac{\pi}{2}\right) = \frac{4(\pi-4)}{\pi^3}$, the right side of (11) follows. The line passing through the point B above and the point $A\left(0, \frac{1}{2}\right)$ has the equation

$$y = \frac{1}{2} + \frac{8-\pi^2}{\pi^3}\cdot x,$$

so (12) follows, as well.

As we have seen in the Introduction, the function $x \to \dfrac{\sin x}{x}$, $0 < x < \dfrac{\pi}{2}$ is strictly decreasing and concave. This implies at once that, similarly, the function $p(x) = \dfrac{\sin \frac{x}{2}}{x}$ is strictly decreasing, and concave, too.

Since

$$p(0) = \lim_{x \to 0} p(x) = \frac{1}{2},$$

$$p'(x) = \left(\frac{1}{2} x \cos \frac{x}{2} - \sin \frac{x}{2}\right) / x^2,$$

and $p\left(\dfrac{\pi}{2}\right) = \dfrac{\sqrt{2}}{\pi}$, in a similar way, as in the proof of Theorem 4 one can deduce the following results:

Theorem 5.

$$\frac{\sqrt{2}}{\pi} \le \frac{\sin \frac{x}{2}}{x} \le \frac{\sqrt{2}}{\pi} + \frac{\sqrt{2}(4-\pi)}{2\pi^2} \left(\frac{\pi}{2} - x\right); \tag{13}$$

$$\frac{1}{2} + \frac{2\sqrt{2} - \pi}{\pi^2} \cdot x \le \frac{\sin \frac{x}{2}}{x} \le \frac{1}{2}. \tag{14}$$

3 New lower and upper bounds for the counterpart of Jordan's inequality

In this section, we will obtain results of type (4) and (5) for the fraction $(\tan x/2)/x$.

First we need an auxiliary result:

Lemma 2. *Let* $f(x) = (\tan x/2)/x$, $0 < x \le \dfrac{\pi}{2}$, $f(0) = \dfrac{1}{2}$. *Then f is a strictly increasing, strictly convex function.*

Proof. We have to prove that $f'(x) > 0$ and $f''(x) > 0$ for $x \in \left(0, \dfrac{\pi}{2}\right)$. After some elementary computations, we get

$$f'(x) = (x - \sin x)/2x^2 \cos^2 x/2 > 0, \text{ as } \sin x < x.$$

For $f''(x)$ one has

$$\left(x^2 \cos^2 \frac{x}{2}\right) f''(x) = (1 - \cos x) - \left(1 - \frac{\sin x}{x}\right)\left(2 \cos \frac{x}{2} - x \tan \frac{x}{2}\right).$$

From the left side of inequality (2) one has

$$1 - \frac{\sin x}{x} < \frac{1 - \cos x}{2},$$

thus by assuming $2 \cos \dfrac{x}{2} - x \tan \dfrac{x}{2} > 0$, by

$$0 < 2 \cos \frac{x}{2} - x \tan \frac{x}{2} < 2 \cos \frac{x}{2} < 2$$

we get

$$\left(x^2 \cos^2 \frac{x}{2}\right) f''(x) > (1 - \cos x) - \left(\frac{1 - \cos x}{2}\right) \cdot 2 = 0,$$

implying $f''(x) > 0$.

On the other hand, for values of x such that eventually

$$2 \cos \frac{x}{2} - x \tan \frac{x}{2} < 0,$$

as $1 - \cos x > 0$, $1 - \dfrac{\sin x}{x} > 0$, clearly $f''(x) > 0$. This proves the strict convexity of f, too. □

Remark 2. As $f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$, by the strict monotonicity of f, a new proof of (6) follows.

Theorem 6. *For all* $0 \le x \le \frac{\pi}{2}$ *one has the double-inequality*

$$\frac{2}{\pi} + \frac{2(\pi - 2)}{\pi^2}\left(x - \frac{\pi}{2}\right) \le \frac{\tan \frac{x}{2}}{x} \le \frac{4 - \pi}{\pi^2} \cdot x + \frac{1}{2}. \tag{15}$$

Proof. As $A\left(0, \frac{1}{2}\right)$, $B\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$ are points on the graph of the convex function f of Lemma 2, we can write that, the graph of f lies below of the segment AB (on $\left[0, \frac{\pi}{2}\right]$). Since the equation of line through AB is

$$y_1(x) = \frac{1}{2} + \frac{4 - \pi}{\pi^2} \cdot x,$$

by $y_1(x) \ge f(x)$, the right side of (15) follows.

The tangent line to the graph of f in the point B has the equation

$$y_2(x) = \frac{2}{\pi} + \frac{2(\pi - 2)}{\pi^2}\left(x - \frac{\pi}{2}\right),$$

and as by convexity of f one has $f(x) \ge y_2(x)$, we get the left side of inequality (15). □

Remark 3. As $\frac{4 - \pi}{\pi^2} + \frac{1}{2} \le \frac{4 - \pi}{\pi^2} \cdot \frac{\pi}{2} + \frac{1}{2} = \frac{2}{\pi}$, the right side of (15) offers an improvement of the Jordan counterpart (6).

Now we will show that the left side of (15) gives an improvement of left side of (7), i.e. a stronger improvement of Kober's inequality will be obtained:

Theorem 7. *One has*

$$\frac{1 - \cos x}{x} \le \frac{2}{\pi} + \frac{2(\pi - 2)}{\pi^2}\left(x - \frac{\pi}{2}\right), \quad 0 < x \le \frac{\pi}{2}. \tag{16}$$

Proof. Replacing x with $\frac{\pi}{2} - x$ in (10), the new inequality becomes (after some elementary computations, which we omit here)

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{\pi - 2}{\pi^2}(\pi - 2x).$$

This is exactly inequality (4) of Section 1, thus relation (16) follows. $\square$

We now state a general form of Theorem 3:

Theorem 8. *Let* $r \in \left(0, \frac{\pi}{2}\right]$. *Then for any* $x \in \left(0, \frac{\pi}{2}\right]$ *one has*

$$\frac{\tan\frac{x}{2}}{x} \le \frac{1}{2} + \frac{\left(\tan\frac{r}{2}\right)/r - \frac{1}{2}}{r}, \quad 0 < x \le r \tag{17}$$

and

$$\frac{\tan\frac{x}{2}}{x} \ge \frac{\tan\frac{r}{2}}{r} + \left(\frac{r - \sin r}{2r^2\cos^2\frac{r}{2}}\right)(x - r) \tag{18}$$

Proof. Apply the same method as in the proof of Theorem 4, by letting

$$B(r, f(r)) = B(r, (\tan r/2)/r)$$

in place of $B(\pi/2, 2/\pi)$. Then (11) and (12) will follow on the basis of computations done in Lemma 1.

Remark 4. For $r = \frac{\pi}{2}$ relation (17) and (18) imply the double-inequality (15).

Inequalities of a different type may be deduced from the following auxiliary result:

Lemma 3. *Put* $A(x) = \frac{2}{x} - \frac{1}{\tan\frac{x}{2}}$, $0 < x \le \frac{\pi}{2}$. *Then A is a strictly increasing, strictly convex function.*

Proof. $A'(x) = \left(x^2 - 4\sin^2\frac{x}{2}\right)/2x^2\sin^2\frac{x}{2} > 0$ by $\sin\frac{x}{2} < \frac{x}{2}$.

$$A''(x) = \left(8\sin^3\frac{x}{2} - x^3\cos\frac{x}{2}\right)/2x^3\sin^3\frac{x}{2} > 0$$

by the known inequality (due to Adamović-Mitrinović, see [3])

$$\frac{\sin t}{t} > \sqrt[3]{\cos t}, \quad 0 < t < \frac{\pi}{2}. \tag{19}$$

Thus A is strictly convex, too. $\square$

We state the following result:

Theorem 9.

$$\frac{1}{2} \le \frac{\tan\frac{x}{2}}{x} \le \frac{1}{2 - \frac{2(4-\pi)}{\pi^2}x^2} \le \frac{1}{2 - x\left(\frac{4-\pi}{\pi}\right)} \le \frac{2}{\pi}, \tag{20}$$

where $0 \le x \le \frac{\pi}{2}$.

Proof. Since $A\left(\frac{\pi}{2}\right) = \frac{4}{\pi} - 1$ and $A(0) := \lim_{x\to 0} A(x) = 0$, and A is convex on $\left[0, \frac{\pi}{2}\right]$, $A(x) \leq \frac{2(4-\pi)}{\pi^2} \cdot x$, so after minor transformation we obtain the second inequality of (20). The other relations of (20) can be verified by taking into account $0 \leq x \leq \frac{\pi}{2}$. □

4 Related inequalities

We now will apply a method of proof of (2) in paper [9]. This will be based on the famous Hermite-Hadamard integral inequality, as well as a generalization obtained by the author in 1982 ([5], see also [6]):

Lemma 4. *Let $f : [a, b] \to \mathbb{R}$ be a continuous convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{21}$$

When f is strictly convex, all inequalities in (13) are strict. When f is concave (strictly concave), then the inequalities in (21) are reversed.

For many applications, refinements, and generalizations of this inequality, see e.g. the book [15].

The following generalization of left side of (21) is due to the author:

Lemma 5. *Suppose that $f : [a, b] \to \mathbb{R}$ is $2k$-times differentiable, and $f^{(2k)}(x) \geq 0$ for all $x \in (a, b)$. Then*

$$\int_a^b f(t)dt \geq \sum_{j=0}^{k-1} \frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!} f^{(2j)}\left(\frac{a+b}{2}\right). \tag{22}$$

When $f^{(2k)}(x) > 0$, the inequality is strict.

Inspired by (22), in 1989 H. Alzer [1] proved the following counterpart:

Lemma 6. *With the same conditions as in Lemma 3, one has*

$$\int_a^b f(t)dt \leq \frac{1}{2}\sum_{i=1}^{2k-1} \frac{(b-a)^i}{i!}[f^{(i-1)}(a) + (-1)^{i-1}f^{(i-1)}(b)]. \tag{23}$$

When $f^{(2k)}(x) > 0$, the inequality is strict.

Remark 5. For a common generalization of (22) and (23), see [8].

Particularly, applying (22) and (23) for $k = 2$, along with (21), we get the following:

Lemma 7. *Suppose that $f : [0, x] \to \mathbb{R}$ is a 4-times differentiable function such that $f''(t) < 0$ and $f^{(4)}(t) > 0$. Then one has the inequalities*

$$f\left(\frac{x}{2}\right) + \frac{x^2}{24}f''\left(\frac{x}{2}\right) < \frac{1}{x}\int_0^x f(t)dt < f\left(\frac{x}{2}\right) \tag{24}$$

and

$$\frac{f(0)+f(x)}{2} \leq \frac{1}{x}\int_0^x f(t)dt$$

$$\leq \frac{f(0)+f(x)}{2} + \frac{x}{4}[f'(0)-f'(x)] + \frac{x^2}{12}[f''(0)+f''(x)]. \tag{25}$$

Proof. Apply Lemma 4 for the concave function f on $[a,b]=[0,x]$. Then the right side of (24) follows. Applying Lemma 5 for $k=2$ we get the inequality

$$\frac{1}{b-a}\int_a^b f(t)dt \geq f\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right), \tag{26}$$

so the left side of (24) follows for $[a,b]=[0,x]$.

In the same manner, applying Lemma 6 for $k=2$, we get

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} + \frac{b-a}{4}[f'(a)-f'(b)]$$

$$+\frac{(b-a)^2}{12}[f''(a)+f''(b)] \tag{27}$$

so (25) will be a consequence of (27), combined with Lemma 4. □

We now are in a position to deduce the following trigonometric inequalities:

Theorem 10. *For all* $0<x<\dfrac{\pi}{2}$ *the following inequalities are true:*

$$\frac{\sin x}{x} < \frac{1-\cos x}{x} < \sin\frac{x}{2}, \tag{28}$$

$$\frac{1}{1+\cos\dfrac{x}{2}} < \frac{\tan\dfrac{x}{2}}{x} < \frac{1}{4}\left(1+1/\cos^2\frac{x}{2}\right). \tag{29}$$

Proof. Apply Lemma 4 for the strictly concave function

$$f(t)=\sin t \text{ for } [a,b]=[0,x].$$

As $\displaystyle\int_0^x \sin t dt = 1-\cos x$, (28) follows.

For the proof of (29), put

$$f(t)=\frac{1}{2\cos^2\dfrac{t}{2}}.$$

Since

$$2f'(t)=\sin\frac{t}{2}\cdot\cos^{-3}\frac{t}{2},$$

$$2f''(t)=\frac{1}{2}\cos^{-2}\frac{t}{2}+\frac{3}{2}\sin^2\frac{t}{2}\cos^{-4}\frac{t}{2}>0,$$

f will be a strictly convex function. As

$$\int_0^x \frac{dt}{2\cos^2 \frac{t}{2}} = \tan \frac{x}{2},$$

by (21) the double inequality (29) follows, by remarking on the left side that

$$2\cos^2 \frac{x}{4} = 1 + \cos \frac{x}{2}.$$

□

Remark 6. As $\sin \frac{x}{2} \leq \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.7$ and $\frac{2}{\pi} \approx 0.63$ the right side of (28) and (7) cannot be compared (for all values of x in $\left[0, \frac{\pi}{2}\right)$). Similarly, the right side of (29) takes the greatest value $\frac{3}{4} = 0.75$ and the least value $\frac{1}{2} = 0.5$, so (29) improves (6) for certain values of x, and vice-versa, (6) is strong than the right side of (29) for other values of x.

Theorem 11. *For all* $0 < x < \frac{\pi}{2}$ *the following hold true:*

$$\left(\sin \frac{x}{2}\right)\left(1 - \frac{x^2}{24}\right) < \frac{1 - \cos x}{x} < \sin \frac{x}{2}, \tag{30}$$

$$\frac{\sin x}{2} < \frac{1 - \cos x}{x} < \frac{\sin x}{2} + \frac{(1 - \cos x)x}{4} - \frac{x^2 \sin x}{12}. \tag{31}$$

Proof. Apply (25) for the same function $f(t) = \sin t$ on $[0, x]$. We omit the details. □

Remark 7. Applying Lemma 6 for the function

$$f(t) = \frac{1}{2\cos^2 \frac{t}{2}},$$

further improvements of type (29) can be deduced.

Remark 8. A version of this paper appeared as a part of our Preprint [18].

Remark 9. For connections of trigonometric and hyperbolic inequalities with the theory of means, see [18], or the recent paper [19].

References

[1] Alzer, H. A note on Hadamard inequalities, *C.R. Math. Rep. Acad. Sci. Canada*, Vol. 11, 1989, 255–258.

[2] Klén, R., M. Visuri, M. Vuorinen. On Jordan type inequalities for hyperbolic functions, *J. Ineq. Appl.*. Vol. 2010, Art. 362548, 14 pages.

[3] Mitrinović, D. S. *Analytic Inequalities*, Springer, 1970.

[4] Neuman, E., J. Sándor. On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities, *Math. Ineq. App.*, Vol. 13, 2010, No. 4, 715–723.

[5] Sándor, J. On Hadamard's inequality, *Mat. Lapok (Cluj)*, Vol. 87, 1982, 427–430 (in Hungarian).

[6] Sándor, J. Some integral inequalities, *Elem. Math. (Basel)*, Vol. 43, 1988, 177–180.

[7] Sándor, J. *Geometric Inequalities*, Edit. Dacia, Cluj (Romania), 1988 (in Hungarian).

[8] Sándor, J. On the Jensen-Hadamard inequality, *Studia Univ. Babeş-Bolyai Math.*, Vol. 36, 1991, 9–15.

[9] Sándor, J. Certain trigonometric inequalities, *Octogon Math. Mag.*, Vol. 9, 2001, No. 1, 331–336.

[10] Sándor, J. A note on the Jordan and Kober inequalities, *Octogon Math. Mag.*, Vol. 13, 2005, No. 1, 404.

[11] Sándor, J. On the concavity of $\sin x$, *Octogon Math. Mag.*, Vol. 13, 2005, No. 1, 406–407.

[12] Sándor, J., M. Bencze, On Huygens' trigonometric inequality, *RGMIA Research Rep. Collection*, Vol. 8, 2005, No. 3, Art. 14.

[13] Sándor, J. A note on certain Jordan type inequalities, *RGMIA Research Rep. Collection*, Vol. 10, 2007, No. 1, Art. 1.

[14] Sándor, J. On $\sin x/x$, *Octogon Math. Mag.*, Vol. 15, 2007, No. 2, 675–677.

[15] Sándor, J. *Selected Chapters of Geometry, Analysis and Number Theory: Classical Topics in New Perspectives*, Lambert Acad. Publ., Germany, 2008.

[16] De Souza, P. N., J.-N. Silva. *Berkeley Problems in Mathematics* (3rd Edition), Springer, 2005.

[17] Nowak, W. G. On fractional part sums: a mean-square asymptotic over short intervals, *Monatsh. Math.*, Vol. 137, 2002, 227–238.

[18] Sándor, J. *Trigonometric and Hyperbolic inequalities*, `http://arxiv.org/abs/1105.0859`.

[19] Sándor, J. On Huygens' inequalities and the theory of means, *J. Math. & Math. Sci.*, Vol. 2012, 2012, Article ID 597490.