Abstract: Let \( P(n) \) be the greatest prime factor of a positive integer \( n \geq 2 \). Let \( L_{\alpha}(n) \) be the number of \( 2 \leq k \leq n \) such that \( P(k) > k^{\alpha} \), where \( 0 < \alpha < 1 \). We prove the following asymptotic formula
\[
L_{\alpha}(n) = (1 - \rho(\alpha))n + O\left(\frac{n}{\log n}\right),
\]
where \( \rho(\alpha) \) is the Dickman’s function.

Keywords: Greatest prime factor, Distribution.


1 Introduction, notation and preliminary results

Let \( P(n) \) be the largest prime factor of a positive integer \( n \geq 2 \). Note that if \( n \) is prime then \( P(n) = n \). Therefore \( 2 \leq P(n) \leq n \) for all \( n \geq 2 \).

Let \( L_{\alpha}(n) \) be the number of \( 2 \leq k \leq n \) such that \( P(k) > k^{\alpha} \), where \( 0 < \alpha < 1 \).

J. Kemeny [1] proved the following asymptotic formula
\[
L_{1/2}(n) = n \log 2 + O\left(\frac{n}{\log n}\right). \tag{1}
\]

Consequently
\[
\lim_{n \to \infty} \frac{L_{1/2}(n)}{n} = \log 2.
\]

That is, the probability or density of the \( n \) such that \( P(n) > \sqrt{n} \) is \( \log 2 \).

Let \( 0 < \alpha < 1 \) a fixed real number. Let \( \epsilon_{\alpha}(x) \) be the set of positive integers not exceeding \( x \) such that in their prime factorization only appear primes \( p \) pertaining to the interval \([0, x^{\alpha}]\). That is, \( \epsilon_{\alpha}(x) \) is the set of positive integers not exceeding \( x \) such that the largest prime factor of these positive integers pertain to the interval \([0, x^{\alpha}]\). We assume that \( 1 \) pertains to the set \( \epsilon_{\alpha}(x) \). These
numbers are called smooth numbers. The number of positive integers pertaining to the set $\epsilon_\alpha(x)$ we denote $N_1(\alpha, x)$. It is well-known [3] the following formula
\[
N_1(\alpha, x) = \rho(\alpha)x + O\left(\frac{x}{\log x}\right). \tag{2}
\]

Therefore, the positive integers in the set $\epsilon_\alpha(x)$ have positive density $\rho(\alpha)$ (this function of $\alpha$ is called Dickman’s function). It is a positive, strictly increasing and continuous function on the interval $(0, 1)$. Besides $0 < \rho(\alpha) < 1$.

Let $\beta_\alpha(x)$ be the set of positive integers not exceeding $x$ such that in their prime factorization appear some prime $p$ pertaining to the interval $(x^\alpha, x]$. That is, $\beta_\alpha(x)$ is the set of positive integers not exceeding $x$ such that the largest prime factor of these positive integers pertain to the interval $(x^\alpha, x]$.

The number of positive integers pertaining to the set $\beta_\alpha(x)$ we denote $N_2(\alpha, x)$.

Note that the sets $\beta_\alpha(x)$ and $\epsilon_\alpha(x)$ are disjoints and $\beta_\alpha(x) \cup \epsilon_\alpha(x) = A$, where $A$ is the set of positive integers $k$ such that $1 \leq k \leq [x]$. Consequently (see (2))
\[
N_2(\alpha, x) = (1 - \rho(\alpha))x + O\left(\frac{x}{\log x}\right). \tag{3}
\]

Let us consider a prime $p$ such that $2 \leq p \leq n$. The set of multiples of $p$ not exceeding $n$ will be denoted $A(n, p)$. Therefore,
\[
A(n, p) = \left\{p.1, p.2, p.3, \ldots, p.\left\lfloor\frac{n}{p}\right\rfloor\right\}. \tag{4}
\]

Let $B_1(n, p)$ be the set of positive integers not exceeding $n$ such that the prime $p$ is their largest prime factor. We denote $B_2(n, p)$ the number of elements in the set $B_1(n, p)$. Note that $B_1(n, p) \subset A(n, p)$. Then,
\[
\sum_{2 \leq p \leq n} B_2(n, p) = n - 1,
\]
\[
N_1(\alpha, n) = 1 + \sum_{2 \leq p \leq n^\alpha} B_2(n, p),
\]
\[
N_2(\alpha, n) = \sum_{n^\alpha < p \leq n} B_2(n, p). \tag{5}
\]

The set of elements $k \in A(n, p)$ such that $p > k^\alpha$ we denote $C_1(n, p)$. The number of elements in the set $C_1(n, p)$ we denote $C_2(n, p)$. Clearly $C_1(n, p) \subset A(n, p)$.

Let $\pi(x)$ be the prime counting function. We need the following Tchebychev’s inequality (see, for example, [2])
\[
\pi(x) < c \frac{x}{\log x}, \tag{6}
\]
where $c$ is a positive constant.
2 Main result

Theorem 2.1 Let $0 < \alpha < 1$. We have the following asymptotic formula

$$L_{\alpha}(n) = (1 - \rho(\alpha))n + O\left(\frac{n}{\log n}\right).$$ (7)

Proof. We have

$$L_{\alpha}(n) = \sum_{2 \leq p \leq n} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0\right) = \sum_{2 \leq p \leq n^\alpha} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0\right) + \sum_{n^\alpha < p \leq n} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0\right).$$ (8)

Let us consider a prime $p$ fixed such that $n^\alpha < p \leq n$.

If $k \in A(n,p)$ then we have $p > n^\alpha \geq k^\alpha$. That is, $p > k^\alpha$. Therefore, $C_1(n,p) = A(n,p)$.

Consequently (see (5) and (3))

$$\sum_{n^\alpha < p \leq n} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0\right) = \sum_{n^\alpha < p \leq n} \left(\sum_{k \in B_1(n,p) \cap A(n,p)} k^0\right) = \sum_{n^\alpha < p \leq n} B_2(n,p) = N_2(\alpha, n) = (1 - \rho(\alpha))n + O\left(\frac{n}{\log n}\right).$$ (9)

Let us consider a prime $p$ fixed such that $2 \leq p \leq n^\alpha$.

Now, let us consider the inequality (where $h$ is a positive integer) $k^\alpha = (p.h)^\alpha \leq p$. This inequality has the solutions

$$h = 1, 2, \ldots, \left[\frac{1 - \alpha}{\alpha}\right].$$

Therefore,

$$C_2(n,p) \leq \left[\frac{p^{1-\alpha}}{\alpha}\right] < p^{1-\alpha} \leq (n^\alpha)^{1-\alpha} = n^{1-\alpha}.$$ (10)

That is,

$$C_2(n,p) \leq n^{1-\alpha}. (10)$$

Now, we have (see (10) and (6))

$$\sum_{2 \leq p \leq n^\alpha} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0\right) \leq \sum_{2 \leq p \leq n^\alpha} \left(\sum_{k \in C_1(n,p)} k^0\right) = \sum_{2 \leq p \leq n^\alpha} C_2(n,p) \leq n^{1-\alpha} \sum_{2 \leq p \leq n^\alpha} 1 = n^{1-\alpha} \pi(n^\alpha) \leq n^{1-\alpha} c \frac{n^\alpha}{\alpha \log n} = c \frac{n}{\alpha \log n}.$$

That is

$$\sum_{2 \leq p \leq n^\alpha} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0\right) = O\left(\frac{n}{\log n}\right).$$ (11)
Equations (8), (9) and (11) give (7). The theorem is proved.
If \( \alpha = 1/2 \) then \( \rho(1/2) = 1 - \log 2 \) (see [3]). In this case, equation (7) becomes the Kemeny’s equation (1).

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References

