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A note on the density of the Greatest Prime Factor

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Abstract: Let $P(n)$ be the greatest prime factor of a positive integer $n \geq 2$. Let $L_\alpha(n)$ be the number of $2 \leq k \leq n$ such that $P(k) > k^\alpha$, where $0 < \alpha < 1$. We prove the following asymptotic formula

$$L_\alpha(n) = (1 - \rho(\alpha))n + O\left(\frac{n}{\log n}\right),$$

where $\rho(\alpha)$ is the Dickman's function.
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1 Introduction, notation and preliminary results

Let $P(n)$ be the largest prime factor of a positive integer $n \geq 2$. Note that if n is prime then $P(n) = n$. Therefore $2 \leq P(n) \leq n$ for all $n \geq 2$.

Let $L_\alpha(n)$ be the number of $2 \leq k \leq n$ such that $P(k) > k^\alpha$, where $0 < \alpha < 1$.

J. Kemeny [1] proved the following asymptotic formula

$$L_{1/2}(n) = n \log 2 + O\left(\frac{n}{\log n}\right). \tag{1}$$

Consequently

$$\lim_{n \to \infty} \frac{L_{1/2}(n)}{n} = \log 2.$$

That is, the probability or density of the n such that $P(n) > \sqrt{n}$ is $\log 2$.

Let $0 < \alpha < 1$ a fixed real number. Let $\epsilon_\alpha(x)$ be the set of positive integers not exceeding x such that in their prime factorization only appear primes p pertaining to the interval $[0, x^\alpha]$. That is, $\epsilon_\alpha(x)$ is the set of positive integers not exceeding x such that the largest prime factor of these positive integers pertain to the interval $[0, x^\alpha]$. We assume that 1 pertains to the set $\epsilon_\alpha(x)$. These numbers are called smooth numbers. The number of positive integers pertaining to the set $\epsilon_\alpha(x)$ we denote $N_1(\alpha, x)$. It is well-known [3] the following formula

$$N_1(\alpha, x) = \rho(\alpha)x + O\left(\frac{x}{\log x}\right). \tag{2}$$

Therefore, the positive integers in the set $\epsilon_\alpha(x)$ have positive density $\rho(\alpha)$ (this function of α is called Dickman's function). It is a positive, strictly increasing and continuous function on the interval $(0, 1)$. Besides $0 < \rho(\alpha) < 1$.

Let $\beta_\alpha(x)$ be the set of positive integers not exceeding x such that in their prime factorization appear some prime p pertaining to the interval $(x^\alpha, x]$. That is, $\beta_\alpha(x)$ is the set of positive integers not exceeding x such that the largest prime factor of these positive integers pertain to the interval $(x^\alpha, x]$.

The number of positive integers pertaining to the set $\beta_\alpha(x)$ we denote $N_2(\alpha, x)$.

Note that the sets $\beta_\alpha(x)$ and $\epsilon_\alpha(x)$ are disjoints and $\beta_\alpha(x) \cup \epsilon_\alpha(x) = A$, where A is the set of positive integers k such that $1 \leq k \leq \lfloor x \rfloor$. Consequently (see (2))

$$N_2(\alpha, x) = (1 - \rho(\alpha))x + O\left(\frac{x}{\log x}\right). \tag{3}$$

Let us consider a prime p such that $2 \leq p \leq n$. The set of multiples of p not exceeding n will be denoted $A(n, p)$. Therefore,

$$A(n, p) = \left\{p.1, p.2, p.3, \ldots, p.\left\lfloor \frac{n}{p} \right\rfloor\right\}. \tag{4}$$

Let $B_1(n, p)$ be the set of positive integers not exceeding n such that the prime p is their largest prime factor. We denote $B_2(n, p)$ the number of elements in the set $B_1(n, p)$. Note that $B_1(n, p) \subset A(n, p)$. Then,

$$\sum_{2 \leq p \leq n} B_2(n, p) = n - 1,$$

$$N_1(\alpha, n) = 1 + \sum_{2 \leq p \leq n^\alpha} B_2(n, p),$$

$$N_2(\alpha, n) = \sum_{n^\alpha < p \leq n} B_2(n, p). \tag{5}$$

The set of elements $k \in A(n, p)$ such that $p > k^\alpha$ we denote $C_1(n, p)$. The number of elements in the set $C_1(n, p)$ we denote $C_2(n, p)$. Clearly $C_1(n, p) \subset A(n, p)$.

Let $\pi(x)$ be the prime counting function. We need the following Tchebychev's inequality (see, for example, [2])

$$\pi(x) < c\frac{x}{\log x}, \tag{6}$$

where c is a positive constant.

2 Main result

Theorem 2.1 *Let $0 < \alpha < 1$. We have the following asymptotic formula*

$$L_\alpha(n) = (1-\rho(\alpha))n + O\left(\frac{n}{\log n}\right). \tag{7}$$

Proof. We have

$$\begin{aligned} L_\alpha(n) &= \sum_{2 \le p \le n} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0 \right) = \sum_{2 \le p \le n^\alpha} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0 \right) \\ &+ \sum_{n^\alpha < p \le n} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0 \right). \end{aligned} \tag{8}$$

Let us consider a prime p fixed such that $n^\alpha < p \le n$.

If $k \in A(n,p)$ then we have $p > n^\alpha \ge k^\alpha$. That is, $p > k^\alpha$. Therefore, $C_1(n,p) = A(n,p)$. Consequently (see (5) and (3))

$$\begin{aligned} & \sum_{n^\alpha < p \le n} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0 \right) = \sum_{n^\alpha < p \le n} \left(\sum_{k \in B_1(n,p) \cap A(n,p)} k^0 \right) \\ &= \sum_{n^\alpha < p \le n} \left(\sum_{k \in B_1(n,p)} k^0 \right) = \sum_{n^\alpha < p \le n} B_2(n,p) = N_2(\alpha, n) \\ &= (1-\rho(\alpha))n + O\left(\frac{n}{\log n}\right). \end{aligned} \tag{9}$$

Let us consider a prime p fixed such that $2 \le p \le n^\alpha$.

Now, let us consider the inequality (where h is a positive integer) $k^\alpha = (p.h)^\alpha \le p$. This inequality has the solutions

$$h = 1, 2, \ldots, \left\lfloor p^{\frac{1-\alpha}{\alpha}} \right\rfloor.$$

Therefore,

$$C_2(n,p) \le \left\lfloor p^{\frac{1-\alpha}{\alpha}} \right\rfloor \le p^{\frac{1-\alpha}{\alpha}} \le (n^\alpha)^{\frac{1-\alpha}{\alpha}} = n^{1-\alpha}.$$

That is,

$$C_2(n,p) \le n^{1-\alpha}. \tag{10}$$

Now, we have (see (10) and (6))

$$\begin{aligned} & \sum_{2 \le p \le n^\alpha} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0 \right) \le \sum_{2 \le p \le n^\alpha} \left(\sum_{k \in C_1(n,p)} k^0 \right) = \sum_{2 \le p \le n^\alpha} C_2(n,p) \\ &\le n^{1-\alpha} \sum_{2 \le p \le n^\alpha} 1 = n^{1-\alpha} \pi(n^\alpha) \le n^{1-\alpha} c \frac{n^\alpha}{\alpha \log n} = \frac{c}{\alpha} \frac{n}{\log n}. \end{aligned}$$

That is

$$\sum_{2 \le p \le n^\alpha} \left(\sum_{k \in B_1(n,p) \cap C_1(n,p)} k^0 \right) = O\left(\frac{n}{\log n}\right). \tag{11}$$

Equations (8), (9) and (11) give (7). The theorem is proved.

If $\alpha = 1/2$ then $\rho(1/2) = 1 - \log 2$ (see [3]). In this case, equation (7) becomes the Kemeny's equation (1).

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