Notes on Number Theory and Discrete Mathematics
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Note on φ, ψ and σ-functions. Part 6

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Abstract: The inequality

$$\varphi(n)\psi(n)\sigma(n) \geq n^3 + n^2 - n - 1.$$

connecting φ, ψ and σ-functions is formulated and proved.
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Let us define for the natural number $n \geq 2$, with canonical representation

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

(where k, α_1, ... $\alpha_k \geq 1$ – natural numbers and $p_1, ..., p_k$ – different prime numbers), the following functions (cf., e.g, [1, 2]):

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1),$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1},$$

$$\Omega(n) = \sum_{i=1}^{k} \alpha_i,$$

$$\underline{set}(n) = \{p_1, ..., p_k\}.$$

Theorem. For every natural number $n \geq 2$

$$\varphi(n)\psi(n)\sigma(n) \geq n^3 + n^2 - n - 1. \tag{1}$$

Proof. Let the natural number n be a prime. Then

$$\varphi(n)\psi(n)\sigma(n) = (n-1)(n+1)^2 = n^3 + n^2 - n - 1$$

and (1) holds.

Let $\Omega(n) = 2$. Then, for n there are two cases.

In the first case, $n = pq$ for two distinct primes p and q. Let $p > q$. Then

$$\varphi(n)\psi(n)\sigma(n) = \varphi(pq)\psi(pq)\sigma(pq) = (p^3 + p^2 - p - 1)(q^3 + q^2 - q - 1)$$

$$= p^3q^3 + p^2q^3 - pq^3 - q^3 + p^3q^2 + p^2q^2 - pq^2 - q^2 - p^3q - p^2q + pq + q - p^3 - p^2 + p + 1$$

$$= p^3q^3 + p^2q^2 - pq - 1 + p^3(q^2 - q - 1) + p^2(q^3 - q - 1) - p(q^3 + q^2 - 2q - 1) - q^3 - q^2 + q + 2$$

(from $p \geq q + 2$)

$$\begin{aligned}&\geq p^3q^3 + p^2q^2 - pq - 1 + p((q+2)^2(q^2 - q - 1) - q^3 - q^2 + 2q + 1)\\ &+ (q+2)^2(q^3 - q - 1) - q^3 - q^2 + q + 2 = p^3q^3 + p^2q^2 - pq - 1\\ &+ p(q^4 + 2q^3 - 2q^2 - 6q - 3) - q^3 - q^2 + 2q + 1) + q^5 + 4q^4 + 2q^3 - 6q^2 - 7q - 2\end{aligned}$$

(from $q \geq 2$)

$$> (pq)^3 + (pq)^2 - pq - 1 = n^3 + n^2 - n - 1$$

i.e., (1) holds, too.

In the second case, $n = p^2$ for a prime number p. Then

$$\varphi(n)\psi(n)\sigma(n) = \varphi(p^2)\psi(p^2)\sigma(p^2) = p(p-1)p(p+1)\frac{p^3 - 1}{p - 1}$$

$$= p^2(p+1)(p^3 - 1) = p^6 + p^5 - p^3 - p^2$$

(from $p \geq 2$)

$$\geq p^6 + 2p^4 - p^3 - p^2 > p^6 + p^4 - p^2 - 1 = n^3 + n^2 - n - 1,$$

i.e., (1) is true.

Let us assume that (1) is valid for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

For p there are two cases. In the first case, $p \notin \underline{set}(n)$. Then

$$\varphi(np)\psi(np)\sigma(np) - (np)^3 - (np)^2 + np + 1$$

$$= \varphi(n)\psi(n)\sigma(n)(p^3 + p^2 - p - 1) - (np)^3 - (np)^2 + np + 1$$

$$\geq (n^3+n^2-n-1)(p^3+p^2-p-1)-(np)^3-(np)^2+np+1$$

$$n^3(p^2-p-1)+n^2(p^3-p-1)-n(p^3+p^2-2p-1)-(p^3+p^2-p-2)$$

(by assumption, $n \geq 4$, but it is enough that $n \geq 2$)

$$\geq 8(p^2-p-1)+4(p^3-p-1)-2(p^3+p^2-2p-1)-(p^3+p^2-p-2)$$

$$= p^3+5p^2-7p-8>0$$

for $p \geq 2$.

In the second case, $p \in \underline{set}(n)$. Then from $\sigma(np) > p\sigma(n)$ we obtain

$$\varphi(np)\psi(np)\sigma(np)-(np)^3-(np)^2+np+1$$

$$> p^3\varphi(n)\psi(n)\sigma(n)-(np)^3-(np)^2+np+1$$

$$> p^3(n^3+n^2-n-1)-(np)^3-(np)^2+np+1$$

$$= p^3n^2-p^3n-p^3-(np)^2+np+1$$

(the smallest value of n is 4)

$$\geq 11p^3-16p^2+4p+1>0$$

for $p \geq 2$.

Therefore, we proved the validity of (1) for the natural number np.

References

[1] Mitrinovic, D., J. Sándor, *Handbook of Number Theory*, Kluwer Academic Publishers, 1996.

[2] Nagell, T., *Introduction to Number Theory*, John Wiley & Sons, New York, 1950.