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Remark on the hollow triangular and quadratic numbers

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Abstract: A new concept related to the n-gonal numbers is introduced and it is illustrated with the cases of triangular and quadratic numbers.
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1 Introduction

The figurate or n-gonal numbers (everywhere the natural number n satisfies the inequality $n \geq 3$) are objects of active research. Here we shall discuss a possible modification of them using examples of modifications of triangular and quadratic numbers, and will study some of their properties.

Each n-gonal number has a countour. Let us call it *hollow n-gonal number.* For example, the fourth triangular number is shown on Fig. 1, while on Fig. 2 its countoure is given, i.e., this is the fourth hollow triangle number.

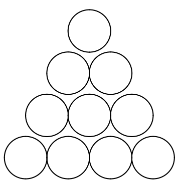

Fig. 1

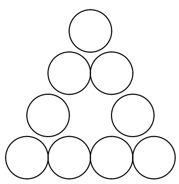

Fig. 2

Let h_s^k be the k-th hollow s-anglular number. Its geometrical interpretation is a figure constructed by circles in the form of right s-gonal figure. It can be easily seen and proved, e.g., by induction, that $h_s^k = s(k-1)$.

Below, we will show that each s-gonal number can be represented as a composition of hollow s-gonal numbers.

First, we note that k-th s-gonal number has the form (e.g., [1, 2])

$$p_s^k = k + \frac{k(k-1)}{2}(s-2).$$

In the particular case when $s = 3$, we obtain the triangular numbers that have the form

$$t_k = p_3^k = k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2}.$$

It can be easy seen that:

$$\begin{array}{ll}
t_1 = h_3^1 & t_6 = h_3^6 + h_3^3 \\
t_2 = h_3^2 & t_7 = h_3^7 + h_3^4 + h_3^1 \\
t_3 = h_3^3 & t_8 = h_3^8 + h_3^5 + h_3^2 \\
t_4 = h_3^4 + h_3^1 & t_9 = h_3^9 + h_3^6 + h_3^3 \\
t_5 = h_3^5 + h_3^2 & t_{10} = h_3^{10} + h_3^7 + h_3^4 + h_3^1
\end{array}$$

etc. More generally, the following assertion is valid.

Theorem 1. Let $k \geq 3$ be a natural number. Then

$$t_k = \sum_{i=0}^{[\frac{k-1}{3}]} h_3^{k-3i}.$$

The recurrent form of the above assertion is given in the following:

Theorem 2. Let $k \geq 4$ be a natural number. Then $t_k = t_{k-3} + h_3^k$.

When $s = 4$ we obtain the quadratic numbers that have the form

$$q_k = p_4^k = k + 2\frac{k(k-1)}{2} = k^2.$$

Now, we can prove:

Theorem 3. Let $k \geq 3$ be a natural number. Then

$$q_k = \sum_{i=0}^{[\frac{k-1}{2}]} h_4^{k-2i}.$$

Theorem 4. Let $k \geq 4$ be a natural number. Then $q_k = q_{k-2} + h_4^k$.

In a next paper, formulas for n-gonal numbers will be discussed.

References

[1] Polygonal Number, *Wolfram MathWorld,* `http://mathworld.wolfram.com/PolygonalNumber.html` (accessed 17 Dec. 2012).

[2] Polygonal number, *Wikipedia, The Free Encyclopedia,* `http://en.wikipedia.org/wiki/Polygonal_number` (accessed 17 Dec. 2012).