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The modular ring Z_5

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Abstract: The characteristics of the modular ring Z_5 are discussed. Each of the five classes is specific for two right-end-digits (REDs), even and odd. This facilitates analysis of the primitive Pythagorean triples, namely, the factors of components and the structure that prevents the two minor components from having REDs of (1,4), (5,6), (5,0) or (9,0). The RED feature is also useful in solving quadratic equations and the quick identification of modular classes. The distribution of powers within Z_5 is complex compared with other modular rings. Even powers are restricted to three Classes for $n = 4r_2 + 2(\bar{2}_4 \subset Z_4)$ but only two Classes when $n = 4r_0\ (\bar{0}_4 \subset Z_4)$. This power distribution is also useful in the analysis of power triples.

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1 Introduction

Modular rings have been shown to be useful for in-depth analyses in many studies in number theory [1] such as Pythagorean triples, power triples ($n > 2$), equations involving primes, Fibonacci and geometric sequences, and infinite series [1, 2, 6].

For modular rings Z_n, n even, the classes contain same-parity integers, whereas with n odd, the classes contain mixed parity integers. For the odd modulo n, only $n = 3$ has been considered. Here we look at $n = 5$. We shall show that the modular ring Z_5 is characterised by right-end-digit (RED) specific classes. We shall also consider related patterns for geometric sequences.

2 The modular ring Z_5

For the decimal system Z_5 might be expected to have some advantages such as each class has specific right-end-digits for even and odd integers. Obviously $\bar{0}_5$ will have REDs of 0 and 5 (Tables 1 and 2).

Row	$f(r)$	$5r_0$	$5r_1+1$	$5r_2+2$	$5r_3+3$	$5r_4+4$
	Class	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{4}_5$
0		0	1	2	3	4
1		5	6	7	8	9
2		10	11	12	13	14
3		15	16	17	18	19
4		20	21	22	23	24
5		25	26	27	28	29
6		30	31	32	33	34
7		35	36	37	38	39
8		40	41	42	43	44
9		45	46	47	48	49
10		50	51	52	53	54

Table 1. Rows of Z_5

Class	REDs of integers N	Rows		Rows for $N \mid 3$	
		Odd N	Even N		
$\bar{0}_5$	0,5	odd	even	$3t$	$t = 1, 2, 3,...$
$\bar{1}_5$	1,6	even	odd	$1+3t$	$t = 0, 1, 2, 3, ...$
$\bar{2}_5$	2,7	odd	even	$2+2t$	$t = 0$
$\bar{3}_5$	3,8	even	odd	$3t$	$t = 0$
$\bar{4}_5$	4,9	odd	even	$1+3t$	$t = 0$

Table 2. Rows of Z_5

In particular, the power structure is quite complex and depends on the class of the power, m, in Z_4 (Table 3). The modular ring Z_4 has four classes: $\bar{0}_4$, $\bar{1}_4$, $\bar{2}_4$, $\bar{3}_4$, $\bar{0}_4\bar{2}_4$ (*even*); $\bar{1}_4\bar{3}_4$ (*odd*), $N = 4r_i + i$ where i is the class. Class $\bar{2}_4$ has no powers and $\bar{3}_4$ has no even powers.

m	Class of $m \in Z_4$		Classes of Z_5 which contain N^m (1 to 9)
1, 5, 9, ...	$\bar{1}_4$	$4r_1+1$	$\bar{0}_5, \bar{1}_5, \bar{2}_5, \bar{3}_5, \bar{4}_5$
2, 6, 10, ...	$\bar{2}_4$	$4r_2+2$	$\bar{0}_5, \bar{1}_5, \bar{4}_5, \bar{4}_5, \bar{1}_5$
3, 7, 11, ...	$\bar{3}_4$	$4r_3+3$	$\bar{0}_5, \bar{1}_5, \bar{2}_5, \bar{3}_5, \bar{4}_5$
4, 8, 12, ...	$\bar{0}_4$	$4r_0$	$\bar{0}_5, \bar{1}_5, \bar{1}_5, \bar{1}_5, \bar{1}_5$

Table 3. Power structure in Z_5

For $m = 4r_3 + 3$, the $\bar{2}_5$ integers when raised to the power m fall in $\bar{3}_5$ whereas $\bar{3}_5$ integers raised to m fall in $\bar{2}_5$. The class of the power is easily identified from the REDs: $2^3 \in \bar{3}_5$ (REDs 3 and 8), whereas $3^3 \in \bar{2}_5$ (REDs 2 and 7). The row structure for odd squares in Z_5 is more complex than in Z_4 as odd squares fall in three classes $\bar{0}_5$, $\bar{1}_5$ and $\bar{4}_5$. However, the rows are still related to the triangular and pentagonal numbers (Table 4).

Class of N^2 (odd)	Row	K	Permitted n*
$\bar{1}_5$	$24K/5$	$½n(3n + 1)$	0, 3, 5, 8
N not divisible by 3		$½n(3n – 1)$	2, 5, 7
N divisible by 3	$4(2 + 9n(n + 1))/5$		1, 3, 6, 8
$\bar{4}_5$	$(24K – 3)/5$	$½n(3n + 1)$	1, 2, 6, 7
N not divisible by 3		$½n(3n – 1)$	3, 8, 9
N divisible by 3	$1 + (36n(n + 1))/5$		4,9 $N \in \bar{2}_5$ $N^* = 7$ 0,5 $N \in \bar{3}_5$ $N^* = 3$
$\bar{0}_5^*$	$(24K + 1)/5$	$½n(3n + 1)$	4, 9
N not divisible by 3		$½n(3n – 1)$	1, 6
N divisible by 3	$(9 + 36n(n + 1))/5$		2, 7

Table 4. Classes $\bar{0}_5, \bar{1}_5, \bar{4}_5$ (*Row/5 = square)

As can be seen from the distribution of the powers, there will be less room for primes in Classes $\bar{1}_5$ and $\bar{4}_5$ so that more primes should occur in Classes $\bar{2}_5$ and $\bar{3}_5$ except in regions where there are few even powers. Since only $\bar{0}_5$ and $\bar{1}_5$ contain

- N^{4m} and $\bar{1}_5 + \bar{1}_5 \in \bar{2}_5$ (no even powers) and
- $\bar{0}_5 + \bar{0}_5 \in \bar{0}_5$ (common factors),

the equation

$$N^{4m} + M^{4m} = Q^{4m} \tag{2.1}$$

must have the class structure

$$\bar{1}_5 + \bar{0}_5 = \bar{1}_5 \qquad \text{(REDs 1 + 0 = 1)}$$

or

$$\bar{0}_5 + \bar{1}_5 = \bar{1}_5 \qquad \text{(REDs 5 + 6 = 1)}$$

These severe constraints lead to the invalidity of Equation (2.1). This has also been illustrated with the modular ring Z_3 [7]. It is of interest to look at the structure [8] of primitive Pythagorean triples (pPts) in Z_5 since 5 is always a factor of one of the components [4].

3 Primitive Pythagorean triples

We have previously shown [4, 5] how integer structure analysis (ISA) in the context of modular rings illustrates how pPts always have 5 as a factor and one of the minor components (x or y) always has 3 as a factor whereas the major component z, cannot have 3 as a factor. A number of classes in Z_5 do not have even powers so that the class structures of pPts in this modular ring are restricted (Table 5).

$(x^2)*$	Class	$(y^2)*$	Class	$(z^2)*$	Class
1	$\overline{1}_5$	0	$\overline{0}_5$	1	$\overline{1}_5$
9	$\overline{4}_5$	6	$\overline{1}_5$	5	$\overline{0}_5$
5	$\overline{0}_5$	4	$\overline{4}_5$	9	$\overline{4}_5$

Table 5. Class constraints on pPt components

The $((x^2)*, (y^2)*)$ RED couples (5,6), (1,4) and (5,0) are excluded because these have incompatible rows [3]. The absence of these RED structures in pPts and that of (9,0) are simply illustrated in Tables 6 and 7.

$(z^2)*$	$z*$ $(a^2 + b^2)*$	$((a^2)*, (b^2)*)$	$x*$ $(a^2 - b^2)*$	$(ab)*$	$y*$ $(2ab)*$	(pPt)*
1	1, 9	(1,0)(5,6)(9,0) (5,4)	1, 9	(1,0)(5,6)(3,0)(5,2) (7,0)	0	101
5	5	(1,4)(5,0)(9,6)	3, 5, 7	(1,2)(1,8)(5,0)(3,6) (7,6)	0, 4, 6	965
9	(3, 7)	(9,4)(1,6)	5	(3,2)(7,2)(1,6)(9,6)	2, 8	549

Table 6. Right-end digits for pPts

(b^2*) \ $(a^2)*$	1	5	9
0	1	5	9
4	5	9	3
6	7	1	5

Table 7. REDs for squares $(a^2 + b^2)*$

As can be seen the main constraint is that z must equal a sum of squares [1] and that odd squares have REDs 1, 5, 9, and even squares have REDs 0, 4, 6. Thus pPt REDs for the squares of x, y, z are 101, 965 and 549.

The Z_5 modular ring has classes that are restricted to particular REDs. Thus for $\overline{0}_5$ all integers have a RED of 0 or 5 so that all integers in $\overline{0}_5$ have a factor of 5. This shows quite simply why one of the pPt components has a factor of 5 (Table 5). The fact that 3 is always a factor of one of the minor components may be shown using the row functions (Table 4) as done for Z_4 [4]. This is simplified because the RED structures of pPts are so limited (Tables 6, 7).

4 Geometric sequences

The Z_5 class patterns for geometric sequences (Table 8) show up certain characteristics of these numbers.

Sequences	**Function**	**Z_5 Class Structure**
Triangular	$\frac{1}{2}n(n+1)$	$\bar{1}_5\ \bar{3}_5\ \bar{1}_5\ \bar{0}_5\ \bar{0}_5$
Tetrahedral	$\frac{1}{6}n(n+1)(n+2)$	$\bar{1}_5\ \bar{4}_5\ \bar{0}_5\ \bar{0}_5\ \bar{0}_5$
Pentagonal	$\frac{1}{3}n(3n+1)$	$\bar{2}_5\ \bar{2}_5\ \bar{0}_5\ \bar{1}_5\ \bar{0}_5$
	$\frac{1}{3}n(3n-1)$	$\bar{1}_5\ \bar{0}_5\ \bar{2}_5\ \bar{2}_5\ \bar{0}_5$
Pyramidal	$\frac{1}{6}n(n+1)(2n+1)$	$\bar{1}_5\ \bar{0}_5\ \bar{4}_5\ \bar{0}_5\ \bar{0}_5$

Table 8. Geometric sequence patterns

The triangular numbers show Class $\bar{3}_5$ in the pattern and pentagonal numbers show Class $\bar{2}_5$ [1, 6]. These two sequences are related to the rows of odd squares. The other sequences show only those classes that contain even powers, that is, $\bar{1}_5, \bar{0}_5, \bar{4}_5$. This suggests that triangular and pentagonal numbers are more "row-related" and the others more directly "square-related".

5 Integer Structure Analysis and quadratic equations

A few examples are given to show how ISA with Z_5 can be used to solve quadratic equations with rational solutions. This method does not involve square roots.

(1) $$6x^2 - x - 22 = 0 \tag{5.1}$$

x must be even and $x^* = 2$ ($x^* = 4$ is too large) so that $x = 5r_2 + 2$. Substituting into (5.1) gives

$$6 \times 25r_2^2 + 115r_2 = 0 \tag{5.2}$$

so that $r_2 = 0$ or $-23/30$, and thus $x = 2$ or $-11/6$.

(2) $$6x^2 - 25x + 21 = 0 \tag{5.3}$$

x will be odd and $x^* = 3$ or 7 to satisfy REDs. 7 is obviously too big so $x = 5r_3 + 3$. Substituting into Equation (5.3) yields

$$30r_3^2 + 11r_3 = 0 \tag{5.4}$$

so that $r_3 = 0$ or $-11/30$, and thus $x = 3$ or $7/6$.

(3) $$5x^2 + 11x - 12 = 0 \tag{5.5}$$

If x is even, the lowest $x^* = 2$, which is too large, so x must be odd. If x is positive, then $x^* = 7$, which is too large, so $x^* = 3$ and $x = -(5r_3 + 3)$. Substitution into (5.5):

$$125r_3^2 + 95r_3 = 0 \tag{5.6}$$

so that $r_3 = 0$ or $-19/25$, and thus $x = -3$ or $4/5$. The advantage of using Z_5 is that this ring is RED specific for each class.

6 Final comments

The simple classification of an integer from the RED has proved useful in some other number theory studies [1, 3]. The modular ring Z_5 is characterised by RED-specific classes. This feature has been shown to be critical for power and factor identification as well as useful in solving quadratic equations. Another example is provided in Table 6, which neatly summarises the RED restrictions on pPts.

The geometric sequences have distinct class patterns and hence RED patterns which are useful in the study of these numbers [6]. For instance, the rows of squares are functions of the triangular and pentagonal numbers, restricted to classes $\overline{0}_5$, $\overline{1}_5$ and $\overline{2}_5$ or $\overline{3}_5$.

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