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On sums of multiple squares

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Abstract: The structural and other characteristics of the Hoppenot multiple square equation are analysed in the context of the modular ring Z_4. This equation yields a left-hand-side and a right-hand-side sum equal to $P_n(24T_n + 1)$ in which P_n, T_n represent the pyramidal and triangular numbers, respectively. This sum always has 5 as a factor. Integer structure analysis is also used to solve some related problems.

Keywords: Integer structure analysis, modular rings, Hoppenot equation, triangular numbers, pentagonal numbers, pyramidal numbers.

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1 Introduction

Hoppenot [2] pointed out that the sum of the squares of $(n + 1)$ consecutive integers, the greatest being $2n(n + 1)$, is equal to the sum of the squares of the next n integers, that is, in notational form

$$\sum_{j=2n^2+n}^{2n^2+2n} j^2 = \sum_{k=2n^2+2n+1}^{2n^2+3n} k^2 \tag{1.1}$$

The first of such a series when $n = 1$ is the Pythagorean triple (3, 4, 5). Thus, the apparent blandness of Hoppenot's observation is misleading as these sums of squares have many interesting features, including their structural characteristics, some of which are developed in this paper.

The underlying pedagogical goals here are:

- knowledge: the conceptual framework of number theory [2];
- attitudes: the inherent attraction of the elegant [3]; and
- skills: notation as a tool of thought [4].

2 Functional *n* characteristics

The initial square on the left-hand-side of Equation (1.1) is given by $(n(2n + 1))^2$ and the sum, S, will be

$$S = P_n(24T_n + 1) \tag{2.1}$$

in which

$$P_n = \frac{1}{6}n(n+1)(2n+1), \tag{2.2}$$

the Pyramidal numbers, and

$$T_n = \frac{1}{2}n(n+1), \tag{2.3}$$

the Triangular numbers [5, 8, 10]. Some examples are illustrated in Table 1.

N	**Left-hand side square**	**Right-hand-side-squares**
1	$3^2 + 4^2 = 25$	$5^2 = 25$
2	$10^2 + 11^2 + 12^2 = 365$	$13^2 + 14^2 = 365$
3	$21^2 + 22^2 + 23^2 + 24^2 = 2030$	$25^2 + 26^2 + 27^2 = 2030$
4	$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 7230$	$41^2 + 42^2 + 43^2 + 44^2 = 7230$
5	$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 19855$	$61^2 + 62^2 + 63^2 + 64^2 + 65^2 = 19855$
6	$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2$ $= 45955$	$85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2 = 45955$
7	$105^2 + 106^2 + 107^2 + 108^2 + 109^2 + 110^2 +$ $111^2 + 112^2 = 94220$	$113^2 + 114^2 + 115^2 + 116^2 + 117^2 + 118^2 +$ $119^2 = 94220$

Table 1. Examples of Hoppenot's equation

Since other Pythagorean triples are embedded within some of the sums, these may be reduced by replacing two squares by one (Table 2). Odd integers of the form ($4r_1 + 1$) may also be sums of squares so that these may also be replaced. For example,

$$2561 = 40^2 + 31^2 = 44^2 + 25^2. \tag{2.4}$$

These squares could be substituted into the right-hand-side of the $n = 4$ square function. Note that the first numbers in each of the left-hand side squares, namely, {3, 10, 21, 36, 55, 78, 105, ...} are elements of the set of coefficients of periodic polynomials [4], and the first numbers in each of the right-hand squares constitute the sequence of centred equation numbers, namely, {5, 13, 25, 41, 61, 85, 113, ...} [10].

n	**Left-hand side square**	**Right-hand-side-squares**
2	$10^2 + 11^2 = 221$	$5^2 + 14^2 = 221$
3	$21^2 + 22^2 + 23^2 = 1454$	$7^2 + 26^2 + 27^2 = 1454$
5	$55^2 + 57^2 + 58^2 + 59^2 = 13119$ $55^2 + 56^2 + 57^2 + 58^2 + 59^2 = 16255$	$11^2 + 33^2 + 62^2 + 63^2 + 64^2 = 13119$ $11^2 + 62^2 + 63^2 + 64^2 + 65^2 = 16255$
6	$78^2 + 79^2 + 81^2 + 82^2 + 83^2 = 32499$	$13^2 + 39^2 + 86^2 + 87^2 + 88^2 + 90^2 = 32499$
7	$105^2 + 106^2 + 107^2 + 109^2 + 110^2 + 111^2$ $= 70012$	$15^2 + 45^2 + 114^2 + 115^2 + 116^2 + 118^2 +$ $119^2 = 70012$

Table 2. Embedding Pythagorean triples from Table 1

The distribution of odd and even squares depends on n (Table 3).

n	**Number of odd terms**	**Number of even terms**
Left-hand side squares		
odd	½ (n + 1)	½ (n + 1)
even	½ n	½ n + 1
Right-hand side squares		
odd	½ (n + 1)	½ (n – 1)
even	½ n	½ n

Table 3. Distribution of even and odd squares

3 Right-end-digit (RED) Analysis

The sum S of the squares always has a factor 5 (cf [9]). This can be illustrated by analysing the REDs (which is essentially the same as working in Z_{10}). With

$$S = f(n).g(n) \tag{3.1}$$

$$f(n) = \tfrac{1}{6}n(n+1)(2n+1)$$

$$g(n) = 12n(n+1)+1$$

the RED of each function is given from n* (where the asterisk indicates the RED) as exemplified in Table 4.

n*	($f(n)$)*	($g(n)$)*	S*
0	5	1	5
1	1	5	5
2	5	3	5
3	4	5	0
4	0	1	0
5	5	1	5
6	1	5	5
7	0	3	0
8	4	5	0
9	5	1	5

Table 4. Right-end-digits (REDs)

Since the product is always 0 or 5, then 5 will always be a factor of S. This can also be proved in Z_5: when

$$n \equiv 0,1,4 \pmod 5, \tag{3.2}$$

then

$$0 \equiv \tfrac{1}{6}n(n+1)(2n+1) \pmod 5, \tag{3.3}$$

and when

$$n \equiv 2,3 \pmod 5, \tag{3.4}$$

then

$$0 \equiv 12n(n+1)+1 \pmod 5. \tag{3.5}$$

4 Difference of squares

The sum of the difference of squares may be used to simplify the arithmetic; that is, the square of each integer may be reduced to simple sums. For example, the first square when $n = 4$ is given by

$$\begin{aligned} 36^2 &= \left(44^2 - 40^2\right) + \left(43^2 - 39^2\right) + \left(42^2 - 38^2\right) + \left(41^2 - 37^2\right) \\ &= 4\left(\left(44 + 40\right) + \left(43 + 39\right) + \left(42 + 38\right) + 41 + 37\right) \\ &= 4 \times 324 \\ &= 1296, \end{aligned} \tag{4.1}$$

or

$$\begin{aligned} 36^2 &= \left(41^2 - 40^2\right) + \left(42^2 - 39^2\right) + \left(43^2 - 38^2\right) + \left(44^2 - 37^2\right) \\ &= 1 \times 9^2 + 3 \times 9^2 + 5 \times 9^2 + 7 \times 9^2. \end{aligned} \tag{4.2}$$

5 Modular ring structure

Integers may be more finely classified (than just even or odd, prime or composite, and so on) by separating them into classes within modular rings [6, 7]. Here, we use the modular ring Z_4 (Table 5), but other modular rings could equally be used.

Row	$f(r)$	$4r_0$	$4r_1 + 1$	$4r_2 + 2$	$4r_3 + 3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 5. Rows of Z_4

Odd squares always fall in Class $\bar{1}_4$ and even squares in Class $\bar{0}_4$, but the sum S can fall in any of the four classes and does so in a regular pattern, $\bar{1}_4\ \bar{1}_4\ \bar{2}_4\ \bar{2}_4\ \bar{3}_4\ \bar{3}_4\ \bar{0}_4\ \bar{0}_4 \ldots$ (Table 6); it is easy to see from the table how this can be generalised.

The squares of integers, N, are well characterised in modular rings [6]: in Z_4 squares only occur in Classes $\bar{0}_4$ (even) and $\bar{1}_4$ (odd). The even integer squares occur in the rows of Table 5 that are also squares. These rows can be reduced to yield the odd-integer rows except when $N = 2^n$ when the square categorisation persists.

- When 3 does not divide odd N, the rows equal $6Q_n$ where $Q_n = \frac{1}{2}n(3n \pm 1)$, the pentagonal numbers, and
- when 3 divides N, the rows equal $(18T_n + 2)$, T_n, the triangular numbers as in Equation (2.3).

n	Class pattern LHS	Class pattern RHS	*S*
1	$\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2=\bar{1}_4+\bar{0}_4=\bar{1}_4$	$\left(\bar{1}_4\right)^2=\bar{1}_4$	$\bar{1}_4$
2	$\left(\bar{2}_4\right)^2+\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2=\bar{0}_4+\bar{1}_4+\bar{0}_4=\bar{1}_4$	$\left(\bar{1}_4\right)^2+\left(\bar{2}_4\right)^2=\bar{1}_4+\bar{0}_4=\bar{1}_4$	$\bar{1}_4$
3	$\left(\bar{1}_4\right)^2+\left(\bar{2}_4\right)^2+\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2=\bar{1}_4+\bar{0}_4+\bar{1}_4+\bar{0}_4=\bar{2}_4$	$\left(\bar{1}\right)^2+\left(\bar{2}_4\right)^2+\left(\bar{3}_4\right)^2=\bar{1}_4+\bar{0}_4+\bar{1}_4=\bar{2}_4$	$\bar{2}_4$
4	$\left(\bar{0}_4\right)^2+\left(\bar{1}_4\right)^2+\left(\bar{2}_4\right)^2+\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2=\bar{0}_4+\bar{1}_4+\bar{0}_4+\bar{1}_4+\bar{0}_4=\bar{2}_4$	$\left(\bar{1}_4\right)^2+\left(\bar{2}_4\right)^2+\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2=\bar{1}_4+\bar{0}_4+\bar{1}_4+\bar{0}_4=\bar{2}_4$	$\bar{2}_4$
5	$\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2+\left(\bar{1}_4\right)^2+\left(\bar{2}_4\right)^2+\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2=\bar{1}_4+\bar{0}_4+\bar{1}_4+\bar{0}_4+\bar{1}_4+\bar{0}_4=\bar{3}_4$	$\left(\bar{1}_4\right)^2+\left(\bar{2}_4\right)^2+\left(\bar{3}_4\right)^2+\left(\bar{0}_4\right)^2+\left(\bar{1}_4\right)^2=\bar{1}_4+\bar{0}_4+\bar{1}_4+\bar{0}_4+\bar{1}_4=\bar{3}_4$	$\bar{3}_4$

Table 6. Class patterns of squares from Table 2

Thus the Hoppenot equation may be broken down structurally to give the basic factors which yield the equality of the two sides of the equation. For example, if E represents even integers and O represents odd integers, then symbolically for $n = 3$:

$$O_1^2+E_1^2+O_2^2+E_2^2=O_3^2+E_3^2+O_4^2 \tag{5.1}$$

where:

- when 3 does not divide O, $O^2=4R_1+1=24Q_n+1$;
- when 3 divides $O, O^2=4R_1+1=9(4T_n+1)$.

Note that when n is odd the leading square is always odd, but when n is even the leading square is even. Consider the numerical example with $n = 3$:

$$21^2+22^2+23^2+24^2=25^2+26^2+27^2. \tag{5.2}$$

Equation (5.2) can be analysed as

$$LHS=4\left(R_1+R_0+R_1^{/}+R_0^{/}\right)+2 \tag{5.3}$$

and

$$RHS=4\left(R_1^{//}+R_0^{//}+R_1^{///}\right)+2 \tag{5.4}$$

Substituting in the row functions:

$$LHS=4\left(2+9n(n+1)+a^2+6Q_n+b^2+6Q_n^{/}\right)+2 \tag{5.5}$$

with $n = 3$, $a = 11$, $Q_n = 22$, $b = 12$, and $Q_n^{/} = 26$.

$$RHS=4\left(2+9m(m+1)+c^2+6Q_n^{//}\right)+2 \tag{5.6}$$

with $m = 3$, $Q_n^{//} = 26$, $c = 13$.

Since 2,4 and 2 cancel out (in that order), on substitution of the appropriate values, we get

$$110+121+132+144 = 156+169+182 = 507 \tag{5.7}$$

or

$$3 \times 13^2 = 3 \times 13^2 \tag{5.8}$$

whereas the sum of squares yields

$$7 \times 29 = 7 \times 29 \tag{5.9}$$

which has no 3,13 factors. The sum-of-squares pathway is different from the structural pathway. Interestingly, the triad {12,13,14] is not a Pythagorean triple.

6 Final comments

Modular rings can be useful in analysing essential structures and for providing new approaches to old equations [6, 7] as in Table 7.

n	***f*(*n*)**	***n***	***f*(*n*)**
$\bar{2}_6$ $(6r_2 - 1)$	$6(R_2 + 3R_4 + r_2)$ $6r_2((6r_2)^2 - 1)$	$\bar{1}_6$ $(6r_1 - 2)$	$6(R_1 + 3R_1^/ + 2r_1 - 2)$ $6r_1(6r_1 - 1)(6r_1 - 2)$
4_6 $(6r_4 + 1)$	$6(R_4 + R_4^/ + 2r_4 + 1)$ $6(6r_4 + 1)(3r_4 + 1)(2r_4 + 1)$	3_6 $(6r_3)$	$6(R_3 + 3R_3^/ + 2r_3)$ $6r_3(6r_3 + 1)(6r_3 + 2)$
$\bar{6}_6$ $(6r_6 + 3)$	$6(R_6 + 3R_6^/ + 2r_6 + 3)$ $6(2r_6 + 1)(3r_6 + 2)(6r_6 + 5)$	$\bar{5}_6$ $(6r_5 + 2)$	$6(R_5 + 3R_1 + 2r_5)$ $6(3r_5 + 1)(2r_5 + 1)(6r_5 + 4)$

Table 7. The modular ring Z_6 [showing that 6|(*n*(*n*+1)(*n*+2))]

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References

[1] Alfred, U. Consecutive Integers with Equal Sums of Squares. *Mathematics Magazine*. Vol. 35, 1962, 155–164.

[2] Ball, W. W. R., H. S. M. Coxeter. *Mathematical Recreations and Essays.* New York: Macmillan, 1956.

[3] Gardner, Howard. *Truth, Beauty and Goodness Reframed: Educating for the Virtues in the Twenty-First Century.* New York: Basic Books, 2011.

[4] Iverson, Kenneth. Notation as a Tool of Thought. *Communications of the ACM*. Vol. 23, 1980, 444–465.

[5] Knopp, Marvin I. (ed.) *Analytic Number Theory: Lecture Notes in Mathematics, Volume 899.* Berlin: Springer, 1981.

[6] Leyendekkers, J. V., A. G. Shannon, J. M. Rybak. *Pattern Recognition: Modular Rings and Integer Structure*. North Sydney: Raffles KvB Monograph No. 9, 2007.

[7] Leyendekkers, J. V., A. G. Shannon. Integer Structure Analysis of the Product of Adjacent Integers and Euler's Extension of Fermat's Last Theorem. *Advanced Studies in Contemporary Mathematics.* Vol. 17, 2008, No. 2: 221–229.

[8] Leyendekkers, J. V., A. G. Shannon. Modular Rings and the Integer 3. *Notes on Number Theory & Discrete Mathematics.* Vol. 17, 2011, No. 2, 47–51.

[9] Leyendekkers, J. V., A. G. Shannon. Why 3 and 5 are always factors of primitive Pythagorean triples. *International Journal of Mathematical Education in Science and Technology.* Vol. 42, 2011, 102–105.

[10] Leyendekkers, J.V., A.G. Shannon. The Structure of Geometrical Number Series. *Notes on Number Theory & Discrete Mathematics*. Vol. 17, 2011, No. 3, 31–37.

[11] Stanton, R.G., D.D. Cowan. Note on a "Square" Functional Equation. *S.I.A.M. Review*. Vol. 12, 1970, 277–279.