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A note on the area and volume of right-angled triangles with integer sides

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Abstract: We show that the area of every right–angled triangle with integer sides is a multiple of 6, and its volume, the product of the lengths of all three sides, is a multiple of 60.
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1 Introduction

The aim of this note is to prove that the area of every right-angled triangle is a multiple of 6, and that the product of the lengths of all three sides, which I define as its "volume", is a multiple of 60. The so–called cosmic triangle $\triangle = (3, 4, 5)$, has the smallest area of 6 squared units, while its volume is 60 cubic units. It therefore follows that every Pythagorean triangle is the "sum" of cosmic triangles. We highlight this fact in **Table 1**, on Pythagorean triples. We begin by quoting Fermat's Little Theorem, as this is integral to our proof.

Lemma 1 (Fermat's Little Theorem). *$a^p - a$ is divisible by p for each prime p and integer a.*

In particular, for primes $p = 3$ and $p = 5$, and any integer a, there exist positive integers $m \equiv m(a)$ and $n \equiv n(a)$, dependent on a, such that

$$a^3 - a = 3m, \quad a^5 - a = 5n \tag{1.1}$$

2 The main result

Let x, y, z be the lengths of the sides of a right-angled triangle, where we assume that $x < y < z$. Then by Pythagoras' Theorem, $x^2 + y^2 = z^2$. If x, y, z are integers, then it is shown in Edwards [1] that there exist positive integers a and b, with $a < b$, such that

$$x = b^2 - a^2, \quad y = 2ab, \quad z = a^2 + b^2. \tag{2.1}$$

Theorem 1 (Main Theorem)**.** *The area of every right-angled triangle with integer sides is a multiple of 6, and the product of all three sides, the volume, is a multiple of 60.*

Proof. The area A and volume V of the triangle $\triangle = (x, y, z)$ with lengths x, y, z are given by

$$A = \frac{1}{2}xy = ab(b^2 - a^2) = ab^3 - ba^3 \tag{2.2}$$

$$V = xyz = 2ab(b^2 - a^2)(b^2 + a^2) = 2(ab^5 - ba^5) \tag{2.3}$$

Importing (1.1) into (2.2) and (2.3), we get

$$A = a(b + 3m) - b(a + 3k) = 3(ma - bk) \tag{2.4}$$

and

$$V = 2a(b + 5n) - 2b(a + 5l) = 10(an - bl) = 2A(a^2 + b^2), \tag{2.5}$$

where m, k, n, l are positive integers. We now show that $ma - bk$ is a multiple of 2. Recall that $b^3 - b = 3m$. If b is odd, then so is b^3, whence $3m$ is even, and hence m is even. The same occurs if b is even. Repeating this on $a^3 - a = 3k$, we easily see that k is also even. Thus from (2.4), the area $A = 3(ma - bk)$ is divisible by primes 2 and 3, and hence is a multiple of 6. Thus for some positive integer Q, $A = 6Q$.
From (2.5), we see that the volume V is a multiple of 10. Also, since $V = 2A(a^2 + b^2)$, and A is divisible by 6, then $\frac{1}{2}V$ is divisible by both 6 and 10, and hence by primes 2, 3, and 5. Thus $\frac{1}{2}V$ is divisible by 30. Therefore V is a multiple of 60. $\square$

Since the $\triangle(3, 4, 5)$ is the smallest with integer sides, with an area of 6 square units and a volume of 60 cubic units, it follows that:

Corollary 1. *Every right-angled triangle with integer sides is the "sum" of the cosmic triangle* $\triangle = (3, 4, 5)$.

Remark 1. *The numbers 4961,6480, 8161 were found on a tablet in Babylon which dates back to 1500 BC, and is a solution of $x^2 + y^2 = z^2$, see Edwards [1], page 4. Surprisingly, the radius of the inscribed circle of this triangle turns out to be 1640, which is about the time (AD 1640) that Fermat proposed his famous problem. Note that each solution to the equation $x^2 + y^2 = z^2$ has a unique radius r, the radius of the inscribed circle. However, two distinct solutions (different triangles) may have the same radius.*

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References

[1] Edwards, H. M., *Fermat's Last Theorem*, Springer Verlag, 1977.

Table 1: Pythagorean triples

The following table generates the Area and Volume of a few primitive solutions to

$$x^2 + y^2 = z^2$$

b	a	x	y	z	$Radius(r)$	$Area$	$Area/6$	$Volume/60(xyz/60)$
2	1	3	4	5	1	6	1	1
3	2	5	12	13	2	30	5	13
4	1	8	15	17	3	60	10	34
4	3	7	24	25	3	84	14	70
5	2	20	21	29	6	210	35	203
5	4	9	40	41	4	180	30	246
6	1	12	35	37	5	210	35	259
6	5	11	60	61	5	330	55	671
7	2	28	45	53	10	630	105	1113
7	4	33	56	65	12	924	124	2002
7	6	13	84	85	6	546	91	1547
8	1	16	63	65	7	504	84	1092
8	3	48	55	73	15	1320	220	3212
8	5	39	80	89	15	1560	260	4628
8	7	15	112	113	7	840	140	3164
...	...	...	...	...	...	...	...	...
121	81	4961	6480	8161	1640	16,073,640	2,676,940	4,372,565,868