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# On certain bounds and limits for prime numbers

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**Abstract:** We will consider various limits and inequalities connected with the $n$-th prime number.
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## 1 Introduction

Let $p$ be the $n$-th prime number. By the famous prime number theorem one has $p_n \sim n \log n$ as $n \to \infty$. An equivalent statement is that, if $\pi(x)$ denotes the number of primes $\leq x$, then $\pi(x) \sim x/\log x$ as $x \to \infty$.

Thus, as a corollary, one gets

$$\frac{p_{n+1}}{p_n} \to 1, \quad \frac{\log p_n}{\log n} \to 1 \text{ as } n \to \infty. \tag{1.1}$$

An old result of Euler states that $\pi(n)/n \to 0$ as $n \to \infty$, thus as by the first relation of (1.1), $\left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)/n} \to 1^0 = 1$, clearly

$$p_{n+1}^{\pi(n)/n} = \left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)/n} \cdot p_n^{\pi(n)/n} \sim p_n^{\pi(n)/n}. \tag{1.2}$$

Put $a_n = p_n^{\pi(n)/n}$. As $\log a_n = \frac{\pi(n)}{n} \cdot \log p_n \sim \frac{\pi(n)}{n} \cdot \log n$, by the second relation of (1.1), so by $\pi(n) \sim \frac{n}{\log n}$, $\log a_n \sim 1$, thus we have deduced the limit:

$$p_n^{\pi(n)/n} \to e \text{ as } n \to \infty. \tag{1.3}$$

In a recent note [8] we have considered the limit:

$$\frac{p_n}{\sqrt[n]{p_1 \cdots p_n}} \to e \text{ as } n \to \infty. \tag{1.4}$$

In what follows, by using certain bounds for $p_1 \dots p_n$ and relations (1.2) and (1.3), we will obtain a new proof of (1.4). Also, we will compare these bounds with certain optimal inequalities connecting $p_1 \dots p_n$ with the binomial coefficient $\binom{n^2}{n}$.

Remarking that the left side of (1.4) could be written also as $\sqrt[n]{x_n}$, where $x_n = p_n^n / p_1 \dots p_n$, one could ask, if the limit of $x_{n+1}/x_n$ does exist? Since $x_{n+1}/x_n = \left(\frac{p_{n+1}}{p_n}\right)^n$, the limit of this sequence would be of interest to study. We will show that, however, this limit doesn't exist, and in fact, one has

$$\liminf_{n\to\infty} \left(\frac{p_{n+1}}{p_n}\right)^n = 1 \tag{1.5}$$

and

$$\limsup_{n\to\infty} \left(\frac{p_{n+1}}{p_n}\right)^n = +\infty. \tag{1.6}$$

Finally, in paper [1] we have introduced the sequence $(\Delta_n(\lambda))_n$

$$\Delta_n = \Delta_n(\lambda) = \binom{n^2}{n} \cdot \exp(-\lambda p_n),$$

where $\lambda > 0$ is a fixed real number; and proved that it is not monotone increasing. We will show here that the sequence $(\Delta_n)$ is an Erdös-Turán type sequence.

# 2 Main results

**Theorem 2.1.** *For $n \geq 10$ one has the double inequality:*

$$e \leq \frac{p_n}{\sqrt[n]{p_1 \cdots p_n}} < \frac{p_n}{p_{n+1}} \cdot p_{n+1}^{\pi(n)/n}. \tag{2.1}$$

**Proof.** Letting $G_n = \sqrt[n]{p_1 \dots p_n}$, in [5] it is proved that for all $n \geq 10$ one has $G_n \leq \frac{1}{e} p_n$. This implies the left side of (2.1). On the other hand, in [6] it is proved that for $n \geq 2$ one has $p_1 \dots p_n > p_{n+1}^{n-\pi(n)}$. After some transformations, this implies the right side of (2.1).

**Corollary 2.1.** *Relation (1.4) holds true.*

**Proof.** By (1.1), (1.2) and (1.3), the right side of (2.1) has limit as $e$. By Theorem 2.1, the limit (1.4) follows.

**Theorem 2.2.** *Relations (1.5) and (1.6) hold true.*

**Proof.** Let $b_n = \frac{p_{n+1} - p_n}{\log p_n}$, $n \geq 1$. Clearly, by (1.1), $b_n \sim \frac{p_{n+1} - p_n}{\log n}$. In 1931 E. Westzynthius [10] proved a famous result:

$$\limsup_{n\to\infty} b_n = +\infty. \tag{2.2}$$

On the other hand, in 2005 (first published in 2009 [3]) D.A. Goldston, J. Pintz and C.Y. Yildirim proved a very famous conjecture, namely that

$$\liminf_{n\to\infty} b_n = 0. \tag{2.3}$$

For earlier, or other properties of the sequence $(b_n)$, see the monograph [4].

Now, as one has

$$\left(\frac{p_{n+1}}{p_n}\right)^n = \left[\left(1+\frac{p_{n+1}-p_n}{p_n}\right)^{p_n/(p_{n+1}-p_n)}\right]^{(p_{n+1}-p_n)n/p_n} = \left[(1+t_n)^{1/t_n}\right]^{q_n},$$

where $t_n \to 0$ by (1.1). On the other hand,

$$q_n = (p_{n+1}-p_n)n/p_n \sim \frac{p_{n+1}-p_n}{\log n} \sim b_n.$$

As $(1+t_n)^{1/t_n} \to e$ as $n \to \infty$, and $e^{+\infty} = +\infty$, $e_0 = 1$, relations (1.5) resp. (1.6) will follow immediately from (2.3), resp. (2.2).

**Theorem 2.3.** *For all $n \geq 10$, one has*

$$\log p_n \geq \frac{1}{n}\log\binom{n^2}{n} + \log\log n + 1 - c_1, \tag{2.4}$$

*where $\binom{n^2}{n}$ denotes a binomial coefficient, and $c_1 = 2.04287\ldots$*

*For all $n \geq 5$ one has*

$$\log p_{n+1} < \left[\frac{1}{n}\log\binom{n^2}{n} + \log\log n - c_0\right] \cdot \frac{1}{1-\dfrac{\pi(n)}{n}}, \tag{2.5}$$

*where $c_0 = 1.10298\ldots$*

**Proof.** In paper [1] the following double inequality is proved:

$$\exp[n(c_0 - \log\log n)] \leq \frac{\binom{n^2}{n}}{p_1 \ldots p_n} \leq \exp[n(c_1 - \log\log n)] \tag{2.6}$$

with optimal constants

$$c_0 = \frac{1}{5}\log 23 + \log\log 5 = 1.10298\ldots$$

and

$$c_1 = \frac{1}{192}\log\binom{36864}{192} + \log\log 192 - \frac{1}{192}\log(p_1 \ldots p_{192}) = 2.04287\ldots$$

Now, combining the right side of (2.6) with the left side of (2.1), relation (2.4) follows.

Remarking that the right side of (2.1) holds for $n \geq 2$, and combining it with the left side of (2.6), relation (2.5) follows.

**Remark 2.1.** As by a result of Rosser-Schoenfeld [7] one has

$$\pi(x) < x/(\log x - 3/2) \text{ for } x > e^{3/2},$$

we get that

$$\frac{1}{1-\pi(n)/n} < \frac{\log n - 3/2}{\log n - 5/2} < 3 \text{ for } n > e^3.$$

Therefore, a more transparent upper bound in (2.5) follows.

A sequence $(\alpha_n)$ is said to be an Erdös-Turán type sequence if, for infinitely many positive integers $k$ one has $\alpha_k < \alpha_{k+1}$ and for infinitely many positive integers $m$ one has $\alpha_m > \alpha_{m+1}$.

**Theorem 2.4.** *Let $\lambda > 0$ be a fixed real number, and define*

$$\Delta_n = \Delta_n(\lambda) = \binom{n^2}{n} \exp(-\lambda p_n).$$

*Then $(\Delta_n)$ is an Erdös-Turán type sequence.*

**Proof.** As $\Delta_k < \Delta_{k+1} \Leftrightarrow \log \Delta_k < \log \Delta_{k+1}$, i.e.

$$\sigma_k := \frac{\log \binom{(k+1)^2}{k+1} - \log \binom{k^2}{k}}{p_{k+1} - p_k} > \lambda \tag{$*$}$$

By the asymptotic formula (see [2], [1])

$$\log \binom{k^2}{k} = \left(k - \frac{1}{2}\right) \log k + k - \frac{1}{2}(1 + \log 2\pi) + O\left(\frac{1}{k}\right)$$

we get

$$\log \binom{(k+1)^2}{k+1} - \log \binom{k^2}{k} = \log k + 2 + O\left(\frac{1}{k}\right). \tag{2.7}$$

By (1.1), (2.7) and (2.3) we get

$$\limsup_{k\to\infty} \sigma_k = +\infty.$$

This means that for any $M > 0$ there exist infinitely many $k$ such that $\sigma_k > M$. Particularly, for $M = \lambda$ inequality $(*)$ holds true. Similarly, by (1.1), (2.7) and (2.2) we get

$$\liminf_{k\to\infty} \sigma_k = 0,$$

which means that for any $a > 0$ there exist infinitely many $m$ such that $\sigma_m < a$. Particularly, for $a = \lambda$, the reverse of inequality $(*)$ holds true. This finishes the proof of Theorem 2.4.

**Remark 2.2.** There are very few monotone sequences connected with primes. One of them is $(p_n/\log n)$, which is strictly increasing, see [9], p. 106.

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