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Note on φ, ψ and σ-functions. Part 4

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Abstract: Two inequalities connecting φ and ψ-functions are formulated and proved.
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1 Introduction

The paper is a continuation of [1, 2, 4]. Here, we formulate and prove two new inequalities connecting φ and ψ-functions.

Let us define for the natural number $n \geq 2$:

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, $\alpha_1, \alpha_2, \ldots \alpha_k \geq 1$ are natural numbers and $p_1, p_2, \ldots, p_k$ are different prime numbers, the following well-known functions (see, e.g. [2, 4])

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1), \text{ and } \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1), \text{ and } \psi(1) = 1,$$

Let us define

$$\underline{cas}(n) = k \text{ and } \underline{cas}(1) = 0,$$

$$\underline{set}(n) = \{p_1, p_2, \ldots, p_k\} \text{ and } \underline{set}(1) = \emptyset.$$

2 Main results

Here we prove the following

Theorem 1. For each natural number $n > 1$:

$$\psi(n) - \varphi(n) \geq \frac{2n}{\underline{max}(n)}.\underline{cas}(n). \tag{1}$$

Proof. Let n be a prime number. Then we check directly, that

$$\psi(n) - \varphi(n) - \frac{2n}{\underline{max}(n)}.\underline{cas}(n) = n + 1 - n + 1 - \frac{2n}{n}.1 = 0.$$

Let us assume that (1) is valid for some natural number $n \geq 2$ and let p be a prime number. Let

$$X \equiv \psi(np) - \varphi(np) - \frac{np}{\underline{max}(np)}.\underline{cas}(np).$$

For p there are two cases.
Case 1: $p \notin \underline{set}(n)$.
If $p < \underline{max}(n)$, then

$$X = \psi(n)(p+1) - \varphi(n)(p-1) - \frac{2np}{\underline{max}(n)}.(\underline{cas}(n) + 1)$$

$$= p(\psi(n) - \varphi(n) - \frac{2n}{\underline{max}(n)}.\underline{cas}(n)) + \psi(n) + \varphi(n) - \frac{2np}{\underline{max}(n)}$$

(by induction assumption)

$$\geq \psi(n) + \varphi(n) - \frac{2np}{\underline{max}(n)}$$

(from inequality

$$\psi(n) + \varphi(n) \geq 2n$$

that is valid for each natural number $n \geq 2$, see, e.g. [3])

$$\geq 2n - \frac{2np}{\underline{max}(n)} > 0.$$

If $p \geq \underline{max}(n)$, then

$$X = \psi(n)(p+1) - \varphi(n)(p-1) - \frac{2np}{p}.(\underline{cas}(n) + 1)$$

$$= p(\psi(n) - \varphi(n) - \frac{2n}{p}.\underline{cas}(n)) + \psi(n) + \varphi(n) - 2n$$

$$\geq p(\psi(n) - \varphi(n) - \frac{2n}{\underline{max}(n)}.\underline{cas}(n)) + \psi(n) + \varphi(n) - 2n$$

(by induction assumption)

$$\geq \psi(n) + \varphi(n) - 2n > 0.$$

Case 2: $p \in \underline{set}(n)$. Then

$$X = \psi(n)p - \varphi(n)p - \frac{2np}{\underline{max}(np)}.\underline{cas}(np)$$

$$= \psi(n)p - \varphi(n)p - \frac{2np}{\underline{max}(n)}.\underline{cas}(n)$$

$$= p(\psi(n) - n - \frac{2n}{\underline{max}(n)}.\underline{cas}(n)) \geq 0$$

by induction assumption. So, Theorem 1 is valid.

Another inequality is valid, too.

Theorem 2. For each natural number $n > 1$:

$$\psi(n) - n \geq \frac{n}{\underline{max}(n)}.\underline{cas}(n). \tag{2}$$

Proof. Let n be a prime number. Then we check directly, that

$$\psi(n) - n - \frac{n}{\underline{max}(n)}.\underline{cas}(n) = n + 1 - n - \frac{n}{n}.1 = 0.$$

Let us assume that (2) is valid for some natural number $n \geq 2$ and let p be a prime number. For p there are two cases.

Case 1: $p \notin \underline{set}(n)$. Let

$$X \equiv \psi(np) - np - \frac{np}{\underline{max}(np)}.\underline{cas}(np).$$

If $p < \underline{max}(n)$, then

$$X = \psi(n)(p+1) - np - \frac{np}{\underline{max}(n)}.(\underline{cas}(n) + 1)$$

$$= p(\psi(n) - n - \frac{n}{\underline{max}(n)}.\underline{cas}(n)) + \psi(n) - \frac{np}{\underline{max}(n)}$$

(by induction assumption)

$$\geq \psi(n) - \frac{np}{\underline{max}(n)}$$

$$\geq \psi(n) - n > 0.$$

If $p \geq \underline{max}(n)$, then

$$X = \psi(n)(p+1) - np - \frac{np}{p}.(\underline{cas}(n) + 1)$$

$$= p(\psi(n) - n - \frac{n}{p}.\underline{cas}(n)) + \psi(n) - n$$

(by induction assumption)

$$\geq \psi(n) - n > 0.$$

Case 2: $p \in \underline{set}(n)$. Then

$$X = \psi(n)p - np - \frac{np}{\underline{max}(np)}.\underline{cas}(np)$$
$$= \psi(n)p - np - \frac{np}{\underline{max}(n)}.\underline{cas}(n)$$
$$= p(\psi(n) - n - \frac{n}{\underline{max}(n)}.\underline{cas}(n)) \geq 0$$

by induction assumption. So, Theorem 2 is valid.

References

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